EIGENVECTOR ROTATION FOR PLANAR ARRAYS

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1. INTRODUCTION

So-called Eigenvector Rotation (EVR), first presented at an IOA conference some years ago [1], is one of the still growing number of high resolution spectrum analysis methods. As with other alternatives to the MUSIC algorithm, there is a class of problems for which it is applicable, and a smaller one for which it is particularly suited and outperforms MUSIC. In this case, it is the resolution of two uncorrelated signals by an unshaded line array of equally spaced elements. This comes about because of the fortuitous existence of an analytic solution for the two signal direction vectors, and because for a $\lambda/2$ line array, delay can be simply derived from phase [2]. In this context, the method has been shown to exhibit robustness to phase and amplitude errors [3]. It has been extended to the case where the signals are correlated [4]. The two signal optimisation has been used as the basis for resolving an arbitrary number of signals [5].

With arbitrary arrays, there is as yet no efficient method of converting phase into delay. Assuming the conversion can be carried out, the angle of arrival is then found by solving a set of non-linear equations. In the following it is shown how to proceed in the case of regularly spaced plane arrays, with a suggestion about how to proceed in the general case.

2. THE THREE STEPS IN EIGENVECTOR ROTATION

Eigenvector rotation can be considered in three steps. The first step, common to all eigenvector-based processing methods, is to estimate the number of signals from the eigenspectrum of the data covariance matrix, and then to establish the bases for the signal and noise subspaces. For white noise, this results in the representation:-

$$\mathbf{R}_{xx} = \sigma^2.\mathbf{I} + \mathbf{W}.\Lambda.\mathbf{W}^H,$$

where the columns of $\mathbf{W}$, $\mathbf{w}_i$ are the eigenvectors corresponding to the greatest eigenvalues. The differences between these and σ^2 form the diagonal matrix Λ. The $\mathbf{w}_i$ form a basis for the signal subspace.

EVR is a signal subspace method. Signal parameters are deduced from the signal eigenvectors of the data covariance matrix, unlike MUSIC which operates on the noise eigenvectors. Any vector $\mathbf{d}$ representing the components of a signal at the array elements can be expressed in the form $\mathbf{d} = \Sigma\, b_i.\mathbf{w}_i$ and if $\mathbf{D}$ is the matrix of all such vectors, $\mathbf{D} = \mathbf{W}.\Lambda^{0.5}.\mathbf{B}$. The vectors $\mathbf{d}_i$ are often referred to as signal direction vectors. Using the singular value decomposition of $\mathbf{R}_{xx}$, it can be shown that $\mathbf{B}$ is unitary[6]. In the case of two signals, $\mathbf{B}$ is generalised from a plane rotation matrix, and depends on a rotation angle and a phase.

The second step in EVR is to determine the optimum **B**. At this point it can be noted that the optimisation does not involve matching the data with postulated signal direction vectors, in contrast to MUSIC and conventional beam-forming. It is still necessary to specify the behaviour which distinguishes direction vectors from arbitrary combinations of the eigenvectors, and with this, an optimality criterion.

Vezzosi and Farrier define direction vectors as those having equal amplitude across the array. They measure deviation from this condition with the so-called Varimax criterion which is the scatter of the squared component amplitudes about the mean. Optimisation consists of finding a rotation matrix **B** which minimises Varimax. When there are two signals, Vezzosi is able to derive closed form expressions for the rotation parameters, involving the solution of a cubic equation.

The third step is to extract angle of arrival from the estimated signal direction vectors. Phase information is available on taking logarithms. This is sufficient if direction of arrival is found by reverting to a beamforming type of operation. If phase can be converted to propagation delay, angle of arrival can be extracted directly. Reconstructing the delays is a process known as phase unwrapping.

3. ANGLE OF ARRIVAL EXTRACTION WITH PLANE ARRAYS.

For an array of n receivers, located at (x_n, y_n), the observed phase ψ_n for a signal of wavelength λ at azimuth and elevation (ϕ, θ) is given by

$$\psi_n = (2\pi/\lambda).(x_n \cos\phi \sin\theta + y_n \sin\phi \sin\theta) - 2k_n.\pi \qquad (1),$$

where k_n is the number of periods. The equation expresses the most general case dealt with here. In (1), the ψ_n are calculated from the EVR estimates of signal direction vectors.

The two unknowns of interest are θ and ϕ. The unknowns k_n are auxiliary parameters. In many applications the elevation angle is also not required. There are more equations than primary unknowns, and a least squares solution is needed, but the auxiliaries must be determined in order to define it completely. Stated thus, the problem falls into the class of mixed integer programming problems. The solution of these is most efficient if features applying to the particular problem are taken into account. Here the approach adopted is to find the k_n's, then apply least squares techniques in one of the ways to be discussed. The unfamiliar part of the problem is in determining the k_n's.

The unwrapping algorithm given by Jeffreys and set out in Appendix A uses the fact that the phase difference between adjacent elements in a line array is nominally constant. A simple extension applies when the spacing varies, since then phase normalised by the corresponding spacing is constant. For regular plane arrays, there are other invariants. For example, the sum of delays at elements in a circular array add to zero. In a triple of elements which form the sides of a right angled triangle, the delays along the two sides including the right angle produce unity if normalised, squared and added. For some more general arrangements of receivers it may be possible to carry out phase unwrapping as discussed in section 6.

To find the angle of arrival, equation (1) may be re-arranged, with co-ordinates expressed in terms of wavelengths, and phase as a number between 0 and 1, to give the equation:-

$$A.u = \varphi \qquad (2),$$

where the elements of the matrix A are

$a_{1n} = x_n/\lambda$, $a_{2n} = y_n/\lambda$,
$u = [\cos\phi \sin\theta, \sin\phi \sin\theta]^T$,
$\varphi = [\psi_n/2\pi + k_n]$.

Equation (2) is linear in u and this opens up three ways to proceed to a solution. It may be found numerically using non-linear least squares techniques. A direct, sub-optimum solution is found if it is assumed that the direction cosines $\cos\phi \sin\theta$ and $\sin\phi \sin\theta$ are independent of each other, in which case the least squares solution is

$$u = (A^TA)^{-1}.A^T.\varphi \qquad (3).$$

If the problem is treated by taking account of the non-linear constraint on u_1 and u_2, a similar expression may be obtained. This is proved in Appendix B and used in the next section.

4. BEARING OF A SIGNAL IN THE PLANE OF A GRID ARRAY

To estimate bearing, time of arrival at every element must be related to time of arrival at the array phase origin. A plane array whose elements are placed at the intersections of two sets of parallel lines offers the same advantages as a line array in phase unwrapping. The line array algorithm depends on finding an estimate of the standard unit of delay along the line. With a plane grid array, phase reconstruction is carried out by finding units of delay along grid lines. An algorithm is given in Appendix A.

If the signal lies in the plane of the array, the dependence on $\sin\theta$ disappears in (1) and the optimum solution must satisfy the condition $\|u(\sigma)\|^2 = 1$. It is shown in Appendix B that this solution is

$$u(\sigma) = (A^TA + \sigma I)^{-1}.A^T.\varphi \qquad (4),$$

where σ is found as the solution of the constraint equation $\|u(\sigma)\|^2 = 1$. It is also shown in the Appendix that the constraint equation is a quartic. This class of equation can be solved using radicals [7].

The squared error of the constrained estimator is $\varphi^T(I - A(A^TA + \sigma I)^{-1}A^T)\varphi - \sigma$, and that of the linear estimator is $\varphi^T(I - A(A^TA)^{-1}A^T)\varphi$. Since equation (1) is satisfied exactly in a noise free environment, (4) must reduce to (3), and from this it follows that σ in some way measures the input noise.

5. BEARING ACCURACY ASSESSMENTS

Bearing accuracy is dependent on the efficiency of the delay reconstruction, and the accuracy of the estimator. With a linear estimator, the accuracy is improved as the number of samples increases, provided that noise characteristics remain the same. In reconstructed delay data, jumps may occur, and the noise properties thus vary. It is of interest to determine the effect of the number of elements and the shape of the array on the accuracy of reconstruction. The latter is significant because for a given number of elements, there are more jumps in the raw phase data, and more chances for a wrong choice of period, if the array is long and thin. In assessing estimation accuracy, only the linear and constrained optimum estimators need be considered. The issue is how effective in reducing error is the extra step of forcing the estimates to obey the constraint.

For the purpose of testing the algorithms, particularly delay reconstruction, simulations were conducted from which statistics were gathered. It is, however, possible, using the expressions above for squared errors to find expressions for some of the bearing error statistics. For the simulation some simplifications to the model were made. The signal was assumed to lie in the plane of the array. The noise was modelled as a uniformly distributed phase deviation about the true delay. It is sufficient to restrict bearings to one quadrant because of array symmetry.

To assess the effect of shape, three arrays of thirty elements in rectangular arrangements of 15x2, 10x3 and 6x5 elements were considered. Unwrapping and estimation errors over 1000 trials were found for a range of bearings and phase deviations, and some results are given below. For the constrained estimator, the parameter was found using Newton-Raphson. The simulations were carried out on a SUN and the program was written in 'C'.

5.1 Unwrapping Efficiency

The simulations showed that the unwrapping algorithm performed successfully at low noise levels on all arrays except at the ends of the quadrant.Tolerance to large phase deviations was dependent on the array. At broadside and endfire, unwrapping failed most often because the sign of the phase gradient was in error. which, however may not lead to large errors at these bearings. Further details are given in Table 1.

Table 1: Unwrapping Failures (%) for three 30 element arrays

Array:	15x2 Angle(deg)				10x 3 Angle(deg)				6x 5 Angle(deg)			
Noise	0	30	60	90	0	30	60	90	0	30	60	90
9	0.170	0.000	0.000	0.728	0.433	0.000	0.000	0.691	0.603	0.000	0.000	0.722
27	0.110	0.000	0.000	0.881	0.429	0.000	0.000	0.686	0.611	0.000	0.000	0.715
45	0.128	0.000	0.011	0.999	0.453	0.000	0.000	0.981	0.606	0.000	0.000	0.833

5.2 Relative Accuracy of the Estimators

Both estimators gave low average bearing error, but the constrained estimator is at an advantage in having lower variance. Typical values are given in table 2. The stopping condition for the constraint iteration was $| \, \|u(\sigma)\|^2 - 1 \, | < 10^{-6}$, and was reached in an average of less than 20 iterations. There was no attempt to investigate a tradeoff between accuracy and time to converge, as a more efficient iteration can be found.

Table 2: Estimator Bearing Accuracy for three 30 element arrays

Array			0	30	60	90
15x2	9	l	0.008(0.070)	-0.006(0.055)	0.003(0.021)	0.000(0.001)
		c	0.006(0.048)	0.000(0.002)	0.001(0.000)	0.000(0.000)
	27	l	0.055(0.684)	0.007(0.477)	-0.001(0.187)	0.001(0.005)
		c	0.009(0.434)	0.001(0.014)	0.001(0.004)	-0.000(0.003)
10x3	9	l	0.002(0.025)	-0.002(0.022)	-0.005(0.010)	-0.000(0.001)
		c	0.004(0.015)	0.000(0.005)	-0.000(0.001)	0.001(0.001)
	27	l	0.008(0.204)	-0.013(0.193)	-0.002(0.090)	-0.003(0.012)
		c	0.009(0.125)	-0.005(0.045)	0.001(0.012)	-0.002(0.007)
6x5	9	l	0.001(0.007)	-0.003(0.010)	-0.005(0.008)	-0.002(0.004)
		c	0.002(0.004)	-0.003(0.010)	-0.003(0.006)	0.001(0.002)
	27	l	0.009(0.067)	-0.002(0.081)	-0.006(0.076)	0.009(0.043)
		c	0.007(0.036)	-0.003(0.077)	-0.003(0.053)	0.004(0.023)

Key: l=linear estimator, c=constrained estimator
average error(variance)

6. DISCUSSION AND CONCLUSIONS

The technique of delay reconstruction for grid arrays is capable of extension in an obvious manner to regularly spaced three-dimensional arrays. For the extension to irregular arrays, a phase-related quantity is required which can be treated similarly to phase difference as above. For triples of elements where the two shorter sides of the triangle are less then $\lambda/2$, estimates of $\sin\phi$ and $\cos\phi$ can be obtained from (1). The circular functions are nominally constant across the array and may serve the purpose.

Overall it may be concluded that the techniques are a viable option for processing of signal phase information for plane arrays at low phase deviations. They are capable of extension as noted.

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7. REFERENCES

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APPENDIX A: PHASE UNWRAPPING ALGORITHMS

A.1 Line Array Spaced At Less Than $\lambda/2$

The algorithm given in [2], for determining the true delays at receivers in a line array given the phase factors p_n, n=1,..,N, where $0 < |p_n| < 1$, is in three steps.

1. Find the differences $\Delta p_m = p_{m+1} - p_m$, m=1,.., N-1, and find the slope of the trend line by first correcting the Δp_m where necessary by adding 1 so that they are all positive. If the average $\Delta p' < 0.5$, the slope of the trend line is positive, otherwise it is negative.

2. Assuming the slope of the trend line is positive, form the true phase differences $\Delta p''_m$ using the preliminary corrections, $\Delta p'_m$.

If $0 < \Delta p'_m < 0.5$, do not adjust further, $\Delta p''_m = \Delta p'_m$. Form the average $\Delta p''$ of these.

For the remaining values, if $\Delta p'_m < \Delta p'' + 0.5$, then $\Delta p''_m = \Delta p'_m$, otherwise $\Delta p''_m = \Delta p'_m - 1$.

3. Starting at receiver 2, add the corrected phase difference to the phase at the previous receiver.

A.2 Grid Array With Closest Neighbours At Less Than $\lambda/2$
Select the array phase origin. This lies at the intersection of two grid lines.

1. For all lines of receivers in one direction, reconstruct the delays as in section A.1.

2. For the line of receivers througth the origin in the other direction, reconstruct delays as before.

3. Along this line, compare the new phases with the old. Where different, adjust the phases of all receivers along the line in the first direction.

APPENDIX B: OPTIMUM CONSTRAINED LEAST SQUARES ESTIMATOR

In equation (2), the squared error for any solution **u** is given by

$$e(u) = (A.u - \varphi)^T(A.u - \varphi),$$

where, because the signal is in the plane of the array, $\|u\|^2 = 1$. To find the optimum solution which takes account of the interdependence of u_1 and u_2, the Lagrange multiplier approach is used.

Consider

$$e(u,\sigma) = (A.u - \varphi)^T(A.u - \varphi) + \sigma(\|u\|^2 - 1.)$$

Differentiating with respect to the components of **u**, the normal equations

$$(A^TA + \sigma I)\, u = A^T \varphi \qquad \text{(B.1)}$$

are found, having the solution $u(\sigma)$ given by equation (4). The parameter σ satisfies

$$\varphi^TA\,(A^TA + \sigma I)^{-2}A^T\varphi = 1 \qquad \text{(B.2).}$$

If g_i are the projections of $A^T\varphi$ onto the eigenvectors of A^TA, and ρ_i. are the eigenvalues, then (B.2) can be written as $g_1^2(\rho_2+\sigma)^2 + g_2^2(\rho_1+\sigma)^2 = (\rho_1+\sigma)^2(\rho_2+\sigma)^2$, which is of degree four.