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EXACT POWER FLOW RELATIONSHIPS BETWEEN MANY MULTI-COUPLED, MULTI-MODAL SUB-SYSTEMS

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ABSTRACT

In a number of recent publications[1, 2, 3, 4] the authors have derived and made use of exact power flow results for a pair of multi-modal sub-systems coupled by a conservative spring at a single point. This work has been used to study the assumptions inherent in traditional Statistical Energy Analysis (SEA) techniques. Although giving many insights, the limitations of the original model have restricted the number of situations where this earlier work could be used. The present paper lays the theoretical groundwork to relax the most severe of these restrictions. Specifically, the theory developed allows for arbitrary numbers of sub-systems and these can be coupled in any fashion desired, including having more than one coupling between pairs of sub-systems. The couplings are still provided by conservative springs at discrete points but it is considered that this is relatively unimportant for studies in a number of areas of interest.

The current work is primarily aimed at quantifying deviations from the mean power flows predicted by SEA where restrictions on coupling details are more than outweighed by having exact results to compare with. Such studies are of particular relevance to SEA models having relatively small numbers of sub-systems or where the couplings are strong. Unfortunately, the results presented are not in closed form, rather an algorithm is derived that is amenable to computer solution. Although not discussed here it is clear that closed form solutions are recoverable for certain standard geometries.

1. INTRODUCTION

Statistical Energy Analysis (SEA) is a technique for studying the flow of energy between coupled sub-systems in terms of ensemble averages taken across realisations of sub-systems with different properties. To enable the technique to be thoroughly studied requires that exact calculations be made for given sets of sub-system parameters, i.e., a deterministic approach. The authors have made such studies for the case of two sub-systems coupled at a single point by a linear spring[1, 2, 3, 4], see for example, Figure 1. The current paper expands this work by considering an arbitrary number of sub-systems coupled by many springs. The springs used are still taken to be linear and apply forces at single points on the sub-systems, see Figure 2. However, no restrictions are placed on their strengths or the properties of the sub-systems, save that they can be represented by Green functions relating harmonic force at any point to response at any other. The Green function approach allows a theory to be derived in its most general form without reference to the details of particular sub-systems. This method has been used by Langley[5] amongst others. The key difference between that work and the present paper is the use only of the Green functions of the *uncoupled* sub-systems. This is of course necessary when actual problems are to be tackled, since these are the only functions normally available.

2. THEORY

Consider N sub-systems, labelled by lower-case letter subscripts and M springs indicated by numbered subscripts. Then label one end of each spring A and the other B, see again Figures 1 and 2. Clearly, $M \geq N-1$ if the sub-systems are to form a single system for study. Next, let the vector $\{Y\}_A$ be the displacements from their mean position of the ends A and $\{Y\}_B$ those at the ends B. Obviously, both

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vectors have M elements. To begin the analysis, assume that all the coupling springs are of zero strength, i.e., the sub-systems are completely uncoupled, and indicate this by a subscript 0. The vectors $\{Y\}_{A0}$ and $\{Y\}_{B0}$ then represent the displacements at the spring attachment points due to external forcing alone. Next, consider the displacement at the point of attachement of one spring on a sub-system caused by deflection of another spring on the *same* sub-system, in the absence of forcing and indicate this by subscript U. This displacement will be governed by the relevant Green function for the sub-system and the spring constant of the deflected spring. If *all* the springs on all the sub-systems deflect this may be written as

$$\{Y\}_{AU} = [A]_A(\{Y\}_{BU} - \{Y\}_{AU}),$$

where $[A]_A$ is a matrix of Green function and spring constant products with one column for each spring and zero elements occurring where sub-systems are not directly connected to each other. When the whole system is externally excited the deflections due to the forcing may be superposed on those due to the couplings to give

$$\{Y\}_A = \{Y\}_{A0} - [A]_A(\{Y\}_A - \{Y\}_B) \tag{1}$$

and of course the same is true for $\{Y\}_B$

$$\{Y\}_B = \{Y\}_{B0} - [A]_B(\{Y\}_B - \{Y\}_A) = \{Y\}_{B0} + [A]_B(\{Y\}_A - \{Y\}_B). \tag{2}$$

Subtracting these two equations gives

$$\{Y\}_A - \{Y\}_B = \{\Delta Y\} = (\{Y\}_{A0} - \{Y\}_{B0}) - ([A]_A + [A]_B)\{\Delta Y\},$$

$$= \{\Delta Y\}_0 - ([A]_A + [A]_B)\{\Delta Y\},$$

where $\{\Delta Y\}$ is the vector of spring compressions and $\{\Delta Y\}_0$ the changes in separation of the spring attachment points in the absence of coupling. This expression may be inverted to yield

$$\{\Delta Y\} = \Big[[I] + [A]_A + [A]_B\Big]^{-1}\{\Delta Y\}_0. \tag{3}$$

$$= [D]^{-1}\{\Delta Y\}_0.$$

where $[D]$ represents the quantity in brackets. Notice that the determinant of $[D]$ is a measure of coupling strength which tends to unity as the coupling becomes weak (weak coupling is usually defined as that where the behaviour of the sub-systems is not greatly affected by that of the coupled sub-systems, i.e., where $[D]$ becomes $[I]$). Also, the matrix $[D]$ has dimensions fixed by the number of springs rather than the number of sub-systems. Assuming that $[D]$ can be formed and inverted to produce $\{\Delta Y\}$, it is possible to express the equations for the individual coupling point motions as

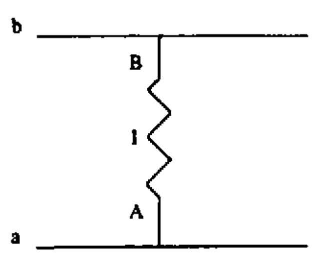

Figure 1 - Two sub-systems and one spring.

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$$\{Y\}_A = \{Y\}_{A0} - [A]_A [D]^{-1} \{\Delta Y\}_0 = \left[([I] - [A]_A [D]^{-1}) \,|\, ([A]_A [D]^{-1}) \right] \begin{Bmatrix} \{Y\}_{A0} \\ - \\ \{Y\}_{B0} \end{Bmatrix}$$

and

$$\{Y\}_B = \{Y\}_{B0} + [A]_B [D]^{-1} \{\Delta Y\}_0 = \left[([A]_B [D]^{-1}) \,|\, ([I] - [A]_B [D]^{-1}) \right] \begin{Bmatrix} \{Y\}_{A0} \\ - \\ \{Y\}_{B0} \end{Bmatrix}.$$

That is, in terms only of the properties of the sub-systems and the responses of the uncoupled problem. Notice that the vector

$$\begin{Bmatrix} \{Y\}_{A0} \\ - \\ \{Y\}_{B0} \end{Bmatrix}$$

has $2M$ elements and describes the motions of all the spring attachement points in the absence of coupling. Finally, the motions of other points within a sub-system can be generated from the vectors $\{Y\}_A$ and $\{Y\}_B$ using the Green functions evaluated for the responses at the points of interest rather than the spring attachement points.

3. ENERGY FLOWS

The previous equations allow all the quantities of interest for a particular case to be determined since energy flows can be found from the product of force and velocity at the ends of the springs, or, in general, from the diagonal elements of $\{Y\}_A^* \{Y\}_B^T$ multiplied by the relevant spring constants (where a $*$ indicates the complex conjugate), i.e., based on

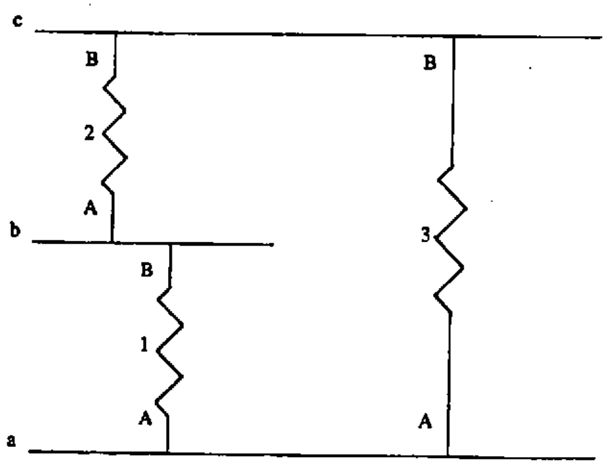

Figure 2 - Three sub-systems and three springs.

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$$\{Y\}_A^*\{Y\}_B^T = \left[(\{I\}-[A]_A[D]^{-1})^* \mid ([A]_A[D]^{-1})^*\right]\begin{Bmatrix}\{Y\}_{A0}^* \\ - \\ \{Y\}_{B0}^*\end{Bmatrix}\left\{\{Y\}_{A0}^T \mid \{Y\}_{B0}^T\right\}$$

$$\times\left[([A]_B[D]^{-1})^T \mid (\{I\}-[A]_B[D]^{-1})^T\right].$$

To determine the vectors $\{Y\}_{A0}$ and $\{Y\}_{B0}$ consider F_a to be a forcing function applied to the a th sub-system and Y_{aj0} the response of the sub-system, when uncoupled, at the point of connection of the j th spring (the particular end, A or B, not being specified). Then

$$Y_{aj0} = \int_a g_a(x_j,x)F_a(x)\mathrm{d}x$$

where x is a suitable dummy variable and the integral is taken over the a th sub-system. Here, $g_a(x_j,x)$ is the Green function for sub-system a evaluated at the point of attachment of spring j due to forcing at the point x. Now if the forcing function is separable in time and space then

$$F_a(x) = F_a \times f_a(x)$$

where $f_a(x)$ is a function only in space so that

$$Y_{aj0} = F_a\int_a g_a(x_j,x)f_a(x)\mathrm{d}x$$

These Green function integrals can be formed into a $2M\times N$ matrix $[gf]$ defined by

$$\begin{Bmatrix}\{Y\}_{A0} \\ - \\ \{Y\}_{B0}\end{Bmatrix} = [gf]\{F\} \tag{4}$$

where $\{F\}$ contains the time varying forcing functions for each of the N sub-systems. Notice that each column of $[gf]$ is specific to a particular sub-system, each row to the end of a spring and that the ordering must align with that chosen for the vector of spring motions. Since one end of a spring can only be connected to one sub-system each row has one and only one element. When the energy flows are considered this matrix formulation leads to

$$\begin{Bmatrix}\{Y\}_{A0}^* \\ - \\ \{Y\}_{B0}^*\end{Bmatrix}\left\{\{Y\}_{A0}^T \mid \{Y\}_{B0}^T\right\} = [gf]^*[S_{FF}][gf]^T \tag{5}$$

where $[S_{FF}]$ is the square matrix of forcing spectra and co-spectra and is diagonal for forcing uncorrelated between the sub-systems. This leads to

$$\{Y\}_A^*\{Y\}_B^T = \left[(\{I\}-[A]_A[D]^{-1})^* \mid ([A]_A[D]^{-1})^*\right][gf]^*[S_{FF}][gf]^T \tag{6}$$

$$\times\left[([A]_B[D]^{-1})^T \mid (\{I\}-[A]_B[D]^{-1})^T\right]$$

and, of course, only the M elements on the diagonal of this matrix need be considered, leading to the M energy flows through the M springs. Notice that for point forcing, where $f(x)=\delta(x_0)$, (where the subscript 0 indicates the point of forcing) the terms in the matrix $[gf]$ are of the form

$$g_a(x_j,x_0).$$

To complete the analysis, the energies flowing into the N sub-systems must be found (the energy dissipated being the difference between these and the flows in the springs). They are found by integrating the products of response and external force over the sub-systems of interest. The responses at arbitrary positions x_{a-N} within the sub-systems are given by

$$\{Y(x)\} = \begin{Bmatrix} Y_a(x_a) \\ Y_b(x_b) \\ \vdots \\ Y_N(x_N) \end{Bmatrix} = \begin{Bmatrix} \int_a g_a(x_a,x)F_a(x)\mathrm{d}x \\ \int_b g_b(x_b,x)F_b(x)\mathrm{d}x \\ \vdots \\ \int_N g_N(x_N,x)F_N(x)\mathrm{d}x \end{Bmatrix} + [B(x)]\{\Delta Y\}$$

where $[B(x)]$ is a non-square matrix of spring constant and Green function products, with the Green functions evaluated *at the points of interest*. It has dimensions $N \times M$ and zero elements in the rows corresponding to springs not directly connected to the relevant sub-systems. The input powers are then derived from the diagonal elements of

$$\int [Y(x)]^*[F(x)]^T\mathrm{d}x.$$

which are functions of the forcing spectra and co-spectra.

4. EXAMPLES

To make use of these relationships the matrices $[A]_A$, $[A]_B$, $[D]$, $[gf]$, $[S_{FF}]$ and $[B(x)]$ must be found from the sub-system properties and the external forcing. Unfortunately, no simple expressions can be given for the elements of these matrices, since they encompass the geometry of the problem which need take no general form (because not all sub-systems are necessarily mutually connected and some may be multiply connected). The matrix $[D]$ however, is simply the sum of $[I]$, $[A]_A$ and $[A]_B$. Also the matrices $[A]_A$, $[A]_B$ and $[D]$ have the same order as the number of springs rather than the number of sub-systems or connection points, which apply to $[gf]$, $[S_{FF}]$ and $[B(x)]$. This leads to a considerable saving in effort when dealing with large, sparsely coupled problems. The form of these various terms is best illustrated by considering simple examples where the forcing is applied at one point per sub-system and is also separable in time and space. Take the case where $N=2$ and $M=1$, i.e., two sub-systems a and b, with a single spring of strength K_1, see again Figure 1. The vectors $\{Y\}_A$ and $\{Y\}_B$ are then scalars

$$\{Y\}_A = Y_{a1A}$$

and

$$\{Y\}_B = Y_{b1B};$$

as are $[A]_A$ and $[A]_B$:

$$[A]_A = K_1(g_a(x_1,x_1)),$$

$$[A]_B = K_1(g_b(x_1,x_1))$$

and $[D]$ becomes

$$[D] = 1 + K_1(g_a(x_1,x_1) + g_b(x_1,x_1)).$$

The matrix $[gf]$ is

$$[gf] = \begin{bmatrix} g_a(x_1,x_0) & 0 \\ 0 & g_b(x_1,x_0) \end{bmatrix}$$

and $[S_{FF}]$ becomes

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$$[S_{FF}] = \begin{bmatrix} S_{FaFa} & S_{FaFb} \\ S_{FbFa} & S_{FbFb} \end{bmatrix}$$

while the matrix $[B(x)]$ is

$$[B(x)] = \begin{bmatrix} K_1 g_a(x_a, x_1) \\ -K_1 g_b(x_b, x_1) \end{bmatrix}.$$

Next consider $N=3$ and $M=3$, i.e., three sub-systems a, b and c with three springs 1, 2 and 3, with two springs connecting each sub-system to the two others, see Figure 2. Here the vectors $\{Y\}_A$ and $\{Y\}_B$ are written as

$$\{Y\}_A = \begin{Bmatrix} Y_{a1A} \\ Y_{a3A} \\ Y_{b2A} \end{Bmatrix}$$

and

$$\{Y\}_B = \begin{Bmatrix} Y_{b1B} \\ Y_{c3B} \\ Y_{c2B} \end{Bmatrix}.$$

$[A]_A$ and $[A]_B$ become

$$[A]_A = \begin{bmatrix} K_1 g_a(x_1,x_1) & K_3 g_a(x_1,x_3) & 0 \\ K_1 g_a(x_3,x_1) & K_3 g_a(x_3,x_3) & 0 \\ -K_1 g_b(x_2,x_1) & 0 & K_2 g_b(x_2,x_2) \end{bmatrix},$$

$$[A]_B = \begin{bmatrix} K_1 g_b(x_1,x_1) & 0 & -K_2 g_b(x_1,x_2) \\ 0 & K_3 g_c(x_3,x_3) & K_2 g_c(x_3,x_2) \\ 0 & K_3 g_c(x_2,x_3) & K_2 g_c(x_2,x_2) \end{bmatrix}$$

with $[D]$ given by

$$\begin{bmatrix} 1+K_1(g_a(x_1,x_1)+g_b(x_1,x_1)) & K_3 g_a(x_1,x_3) & -K_2(g_b x_1,x_2) \\ K_1 g_a(x_3,x_1) & 1+K_3(g_a(x_3,x_3)+g_c(x_3,x_3)) & K_2 g_c(x_3,x_2) \\ -K_1 g_b(x_2,x_1) & K_3 g_c(x_2,x_3) & 1+K_2(g_b(x_2,x_2)+g_c(x_2,x_2)) \end{bmatrix}.$$

Writing α_{aji} for $K_i g_a(x_j,x_i)$ the structure of this matrix becomes more apparent, viz.,

$$[D] = \begin{bmatrix} 1+\alpha_{a11}+\alpha_{b11} & \alpha_{a13} & -\alpha_{b12} \\ \alpha_{a31} & 1+\alpha_{a33}+\alpha_{c33} & \alpha_{c32} \\ -\alpha_{b21} & \alpha_{c23} & 1+\alpha_{b22}+\alpha_{c22} \end{bmatrix}.$$

For this system the matrix $[gf]$ is

$$[gf] = \begin{bmatrix} g_a(x_1,x_0) & 0 & 0 \\ g_a(x_3,x_0) & 0 & 0 \\ 0 & g_b(x_2,x_0) & 0 \\ 0 & g_b(x_1,x_0) & 0 \\ 0 & 0 & g_c(x_3,x_0) \\ 0 & 0 & g_c(x_2,x_0) \end{bmatrix}.$$

and $[S_{FF}]$ is given by

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$$[S_{FF}] = \begin{bmatrix} S_{FaFa} & S_{FaFb} & S_{FaFc} \\ S_{FbFa} & S_{FbFb} & S_{FbFc} \\ S_{FcFa} & S_{FcFb} & S_{FcFc} \end{bmatrix}.$$

Finally, $[B(x)]$ becomes

$$[B(x)] = \begin{bmatrix} K_1 g_a(x_a, x_1) & 0 & K_3 g_a(x_a, x_3) \\ -K_1 g_b(x_b, x_1) & K_2 g_b(x_b, x_2) & 0 \\ 0 & -K_2 g_c(x_c, x_2) & -K_3 g_c(x_c, x_3) \end{bmatrix}.$$

Notice that for the matrices $[A]_A$, $[A]_B$ and $[D]$, each column relates to a particular spring in the problem and that when a spring is omitted a complete column may be removed since its elements become unity on the leading diagonal and zero elsewhere. For the matrices $[gf]$ and $[B(x)]$ each column is for a given sub-system and each row a spring end, while for $[S_{FF}]$ the rows and columns are each for a sub-system. It is clear that if the system of interest exhibits symmetry of any kind then suitable labelling of the sub-systems and springs will produce matrices which show similar features and these may lead to simplified solutions, etc.

5. ALGORITHM

The following algorithm for calculating energy flows may be deduced from the preceding analysis assuming that the forcing is separable in time and space :-

(1) Label all the sub-systems and springs, indicating for each spring an end A and an end B, as illustrated in Figures 1 and 2.

(2) Chose a frequency of interest and use the Green functions of the uncoupled sub-systems to form the matrices $[A]_A$ and $[A]_B$; equations (1) and (2).

(3) Sum $[A]_A$ and $[A]_B$ with [I] to form $[D]$ and invert it; equation (3).

(4) Use the chosen forcing model and the sub-system Green functions to form the matrices $[gf]$ and $[S_{FF}]$; equations (4) and (5).

(5) Calculate the *diagonal* elements of the energy flow matrices; equation (6).

(6) Deduce the dissipation energy flows using the inputs and flows through the springs.

(7) Assuming that the kinetic energies of the sub-systems are related directly to the dissipation flows, find the sub-system energy levels (i.e., assuming viscous damping).

(8) Ratio the energy levels and coupling energy flows to generate the coupling loss factors used by SEA.

(9) Repeat steps 2 to 8 for all the frequencies of interest.

(10) If total energy flows are required, integrate the flows over the frequency range of interest, noting that a matrix inversion is required for each frequency examined, before taking steps 7 and 8.

This scheme can be considerably speeded up if the matrix $[D]$ can be inverted algebraically, either because the system has some simplifying property, or by using an algebraic manipulator.

6. CONCLUSIONS

This paper has outlined a method for calculating exactly the energy flows between many multi-coupled, multi-modal sub-systems. The method used is couched in terms of Green functions describing the sub-systems, spring constants for the coupling elements and spectral densities for the energy flows. Two simple examples have been given to illustrate the equations derived. The material presented will form the basis of further studies into variance estimates for SEA methods using Monte-Carlo methods. That is, large numbers of randomly varied systems will be formed, this method applied to each in turn and the statistics of the resulting energy flow averages calculated *without* making assumptions concerning coupling strengths, numbers of sub-systems, modal densities, etc.

7. REFERENCES

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