GENERATION OF SHEAR WAVES IN AN INHOMOGENEOUS SEDIMENT

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ABSTRACT

This paper considers an inhomogeneous solid sediment, whose density varies continuously with depth, lying between two homogeneous media - an upper fluid layer representing the ocean and a semi-infinite, homogeneous, solid substrate. The problem considered is that of determining the reflection coefficient of a plane wave incident on the sediment from above. It is assumed that the shear modulus in the sediment is small compared with the bulk modulus. Under these circumstances the resulting equations, governing the generation of shear and compression waves, can be tackled analytically. In particular, the equations demonstrate how a density gradient in the sediment results in the continuous generation of shear waves within the bulk of the solid, and not just at the interfaces between two media. Analytical solutions are derived for the case of an isovelocity sediment, and are used to investigate the effect of a continuous density variation within the sediment, on the reflection of an incoming plane wave from the upper layer.

1. INTRODUCTION

The purpose of this study is to examine the effect on reflection loss of density variation in marine sediments supporting shear waves. The effect of density variation on wave propagation through solids has received relatively little attention in previous published work. Thus, while the equations for wave motion in inhomogeneous solids have been considered by a number of authors [1, 2, 3], most attention has been paid to the general case where shear speed, sound speed and density all vary simultaneously. The resulting equations are complex, and do not lend themselves to easy physical interpretation. Also, solutions are obtainable only for a limited number of circumstances. Hook [1], for example, looks for forms of density and Lamé parameter variation which permit the equations for compression and shear components to be decoupled, but assumes that the Lamé parameters and density all have a similar dependence on depth, while the simpler models of solid sediments developed by other authors [4, 5] do not consider density variations. Density effects <u>have</u> been considered by some authors (see, for example, refs. 6, 7, 8), but these studies have concentrated on fluid sediments only. In a study of the effect of density and speed profile shapes on reflection loss in a fluid layer, Robins [7] concluded that density profile effects can be significant at low frequencies. It was therefore thought worthwhile to extend the analysis of ref. 7 in order to look for analytical solutions for an inhomogeneous solid sediment, and to assess the influence of the density profile in the presence of shear waves.

The following sections show that, provided certain simplifying assumptions are made, the equations of motion can be expressed in a form sufficiently simple to permit analytical treatment. The resulting solutions are used to investigate the importance of density profile effects at various frequencies. It would of course be possible to look at these effects using a numerical model such as SAFARI [9], which represents density variation by splitting the sediment into a number of homogeneous sublayers. However, this method suffers from the disadvantage that it is not easy to differentiate between discretisation effects and the real effects of varying the profile, and Ref. 7 shows that the two effects can be of similar magnitude. For this reason it was considered desirable to pursue the analytical route described below.

2. EQUATIONS FOR SHEAR AND COMPRESSION WAVES IN AN INHOMOGENEOUS MEDIUM

The starting point for the analysis is the linearised momentum equation [1] governing wave propagation in an inhomogeneous medium:

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = \nabla \left[(\lambda + 2\mu) \operatorname{div} \underline{u} \right] - \operatorname{curl} (\mu \operatorname{curl} \underline{u}) - 2 (\nabla \mu) \operatorname{div} \underline{u} + 2 (\nabla \mu \cdot \nabla) \underline{u} + 2 \nabla \mu \wedge \operatorname{curl} \underline{u}, \qquad (2.1)$$

where $\underline{u}$ is the particle displacement vector. In this equation the Lamé parameters λ and μ, and the density ρ, may all be functions of position. We now make a number of assumptions, namely:

- the shear modulus μ is small in relation to the bulk modulus. This is equivalent to the assertion that v^2/c^2 is small, where v is the shear wave speed and c the compression wave speed. [NB This assumption is used only for an inhomogeneous medium. In the homogeneous case it is not necessary to assume small μ].
- the shear wave speed is a slowly varying function of position. This implies that terms involving the gradient of the shear modulus may be neglected.
- the density is a general function of position, and is not constrained to vary slowly. Terms involving the density gradient are therefore retained.

It is shown below that these assumptions lead to relatively simple equations for P and S waves, permitting exact solutions for plane waves. First of all, neglect of the $\nabla\mu$ terms gives

$$\frac{\partial^2 \underline{u}}{\partial t^2} = \frac{1}{\rho} \nabla \cdot \left(\rho c^2 \operatorname{div} \underline{u} \right) - \frac{\mu}{\rho} \operatorname{curl} \operatorname{curl} \underline{u}. \qquad (2.2)$$

Defining $D = \operatorname{div} \underline{u}$, then taking the divergence of the above equation and neglecting $\nabla(\mu/\rho)$ gives

$$\frac{\partial^2 D}{\partial t^2} = \operatorname{div} \left[\frac{1}{\rho} \nabla \left(\rho c^2 D \right) \right]. \qquad (2.3)$$

Noting that the pressure p is defined by $p = -(\lambda + 2\mu/3) D = -\kappa D$, (2.4)

and neglecting terms of order v^2/c^2, results in the equation for p:

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$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\frac{1}{\rho}\nabla\rho \cdot \nabla p. \tag{2.5}$$

This is the well-known equation for pressure waves in an inhomogeneous fluid. The pressure equation is therefore unaffected, to first order, by the presence of shear waves.
We now seek an equation for the shear component of the motion, and assume that the shear wave is polarised in the (x,z) plane. The particle displacement $\underline{u}$ is written in the familiar Helmholtz form

$$\underline{u} = \nabla\phi + \mathrm{curl}\,\underline{\psi}, \tag{2.6}$$

where $\underline{\psi} = (0, \psi, 0)$ has only one non-zero component. To get the equation for $\underline{\psi}$, (2.2) is first re-written as

$$\frac{\partial^2 \underline{u}}{\partial t^2} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \underline{u}. \tag{2.7}$$

Taking the curl of this equation, and noting that curl $\underline{u} = -\nabla^2\underline{\psi}$, gives

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left(\nabla^2\underline{\psi}\right) - \nabla^4\underline{\psi} = -\frac{1}{(\rho v)^2}\nabla\rho \wedge \nabla p. \tag{2.8}$$

This fourth order equation differs from the more familiar Helmholtz type of equation for shear waves in homogeneous media [9] due to the presence of the forcing term on the right hand side, representing the continuous conversion of pressure to shear waves within the medium as a result of the density gradient. The mechanism can easily be understood by considering the propagation of a plane compression wave through a medium in which density varies with depth. Except at normal incidence, the wave front will not lie on a plane of constant ρ. Thus different particle motions will be generated at different positions along the wavefront. That is, a shearing motion is generated, resulting in the production of a shear wave. In the case of normal incidence the motion is of course uniform along the wavefront and so no shear is generated. [The forcing term on the RHS of (2.8) is exactly zero in this case since $\nabla\rho$ and ∇p are parallel]. Note that, in the limit of zero shear strength, equation (2.8) shows that $\underline{\psi}$ satisfies a Poisson equation and is <u>not</u> identically zero. That is, the fluid motion is not irrotational. However, no shear wave can be sustained in that case.
The scalar potential ϕ in (2.6) is related to the pressure via (2.4), namely

$$p = -\kappa\,\mathrm{div}\,\underline{u} = -\kappa\nabla^2\phi. \tag{2.9}$$

If we now make the usual assumption of an exp(-iωt) time dependence, the equations for p and $\underline{\psi}$ become

$$\left(\nabla^2 + k_p^2\right)p = \frac{1}{\rho}\nabla\rho \cdot \nabla p, \tag{2.10}$$

$$\left(\nabla^2 + k_s^2\right) \nabla^2 \underline{\psi} = \frac{1}{(\rho v)^2} \nabla \rho \wedge \nabla p, \tag{2.11}$$

where $k_p = \omega/c$ and $k_s = \omega/v$.

We now introduce new variables in place of ρ and p, which transform the above pair of equations into a form amenable to analytical treatment for a suitably chosen class of density profiles. In an earlier paper on plane wave propagation in an inhomogeneous fluid [10], it was found convenient to work in terms of the variables

$$w = 1/\sqrt{\rho}\,,\ q = p/\sqrt{\rho}, \tag{2.12}$$

in place of ρ and p. These variables also turn out to be suitable for the inhomogeneous solid. Substitution of w and q into equations (2.10), (2.11) leads to the transformed equations

$$\left(\nabla^2 + k_p^2\right) q = \frac{q}{w} \nabla^2 w, \tag{2.13}$$

$$\left(\nabla^2 + k_s^2\right) \nabla^2 \underline{\psi} = -\frac{2}{v^2} \nabla w \wedge \nabla q. \tag{2.14}$$

Note that the RHS of the $\underline{\psi}$ equation now contains a linear dependence on w, in contrast to the nonlinear density dependence in (2.11). If we now assume a plane wave, so that the x dependence of q and $\underline{\psi}$ is represented by $\exp(ik_x x)$, and also assume that ρ is a function only of the vertical coordinate z, then the above equations become

$$\frac{d^2 q}{dz^2} + \left(k_p^2 - k_x^2 - \frac{w''}{w}\right) q = 0, \tag{2.15}$$

$$\left(\frac{d^2}{dz^2} + k_s^2 - k_x^2\right)\left(\frac{d^2}{dz^2} - k_x^2\right) \psi = -\frac{2ik_x}{v^2} q w', \tag{2.16}$$

where a prime denotes differentiation with respect to z.

3. DENSITY PROFILES

Robins [10] has considered forms of density profile for which exact solutions of the Helmholtz equation can be found in an inhomogeneous fluid. Clearly the density profile leading to the simplest form of solution of (2.15) will be one for which w"/w is a constant. This leads to an exponential form for w(z), and hence gives a density profile of the form

$$\rho = A e^{\alpha z}/(e^{\alpha z} + a)^2 = \frac{A}{4a}\,\mathrm{sech}^2\left[\frac{1}{2}(\alpha z - \log a)\right] \qquad (3.1)$$

This is denoted a "sech squared" profile by Robins [10]. It is demonstrated in ref. 10 that this form of profile is capable of giving a good fit to measured profiles in marine sediments [11]. The presence of the three parameters a, A and α in (3.1) means that a family of density profiles of different shapes can be generated by varying the value of a, and then choosing appropriate values of A and α [which are dependent on the chosen 'a' value] to force the density to take pre-determined values at the top and bottom of the inhomogeneous layer. The case $a = 0$ gives the simple exponential profile considered by Tolstoy [12].

It is clear that, if we restrict our attention to an isovelocity layer whose density varies according to (3.1), then (2.15) yields a simple exponential form for q, and likewise the solution of (2.16) is expressible as a sum of exponentials provided the shear speed is constant. Exact solutions are also obtainable, in principle, for a medium whose sound speed varies with depth, but the analysis for such cases is much more complex and has not been pursued in this paper, which is concerned solely with examining density profile effects. This means of course that refraction effects in the sediment are not accounted for, and these often play a dominant role in underwater applications because of the low grazing angles encountered in practice. However, previous work by the author on fluid sediments [7] indicates that, in the frequency range for which the influence of the density profile is important, the effects are of comparable magnitude in refracting and non-refracting media. We might therefore expect any conclusions drawn from this study to be applicable in the refracting case.

4. THREE LAYER MODEL

The solutions outlined in the previous section are used in a three layer model of ocean, sediment and substrate. The upper layer is treated as a lossless fluid, the sediment as an inhomogeneous, isovelocity solid and the substrate layer as a homogeneous solid. Losses are permitted in both sediment and substrate. The sound field in the upper layer is composed of a downgoing incident wave of unit amplitude and a reflected wave of amplitude $|R|$, where R is the reflection coefficient. Transmitted P and S waves are present in the substrate, while the sediment layer contains both downward and upward P and S waves. Thus in all there are seven wave components generated by the incident wave. Their magnitudes are determined by applying the usual matching conditions at the sea/sediment and sediment/substrate interfaces, namely continuous normal velocity and normal stress at both interfaces, continuous tangential velocity and tangential stress at the lower interface, and zero tangential stress at the upper interface. The resulting equations are solved numerically for any prescribed grazing angle and wavenumber of the incident wave.

Numerical results are presented in the following section for a sediment whose density properties are typical of a terrigenous deep sea sediment [11]. The sediment and substrate properties used are shown in Table 1 below. Sound speeds and densities are expressed as dimensionless values, normalised with respect to water density and sound speed. The two density profiles considered [given by $a = 0$ and $a = 0.519$ in equation (3.1)] are plotted in Fig. 1. The '$a = 0$' profile is a simple exponential, while the '$a = 0.519$' profile reproduces realistically the shape of a sediment density profile as given in ref. 11.

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Density at top of sediment	=	1.457
Density at bottom of sediment	=	2.181
Density in substrate	=	2.419
Sediment sound speed	=	0.979
Substrate sound speed	=	2.267
Shear speed/sound speed in sediment	=	0.2
Shear speed/sound speed in substrate	=	0.5
Pressure loss in sediment	=	0.1 dB/wavelength
Pressure loss in substrate	=	0.1 dB/wavelength
Shear loss in sediment	=	1.0 dB/wavelength
Shear loss in substrate	=	0.2 dB/wavelength

Table 1 - Sediment and Substrate Properties

5. NUMERICAL RESULTS

All reflection loss results in this section are presented as graphs of $|R|$ versus grazing angle, at fixed values of the dimensionless frequency given by $k_0h = \omega h/c_0$, where h is sediment thickness and c_0 is the sound speed in the upper (water) layer. Evaluation of reflection loss for a range of dimensionless frequencies shows a significant dependence on profile shape at low frequency, rapidly decreasing as frequency increases. Figures 2 and 3 compare reflection loss for the two sediment density profiles of Fig. 1, at dimensionless frequencies of 2 and 8 respectively. We see from Fig. 2 that there are significant differences, when $k_0h = 2$, at grazing angles above the critical angle for shear wave generation in the substrate (28.1 degrees). The maximum difference is about 8dB. At this frequency the wavelength of sediment shear waves is the same order of magnitude as the sediment thickness. In contrast, the effect of density profile shape is much reduced at the higher frequency $k_0h = 8$ (Fig. 3). The decreasing importance of density profile shape with increasing frequency is reflected in the coefficient of q in equation (2.15), which becomes dominated by the terms proportional to ω^2 as frequency increases. In Fig. 3 the range of grazing angle over which differences are apparent is much reduced, with a maximum difference not exceeding 4dB. This behaviour is similar to that observed with a fluid sediment and substrate [7], where it was concluded that the density profile shape <u>can</u> have a significant effect, but only in the frequency range where wavelengths are comparable with the thickness of the layer. Thus differences in density gradient are important over a limited frequency range. However, it is not permissible to ignore the density gradient completely - setting the density gradient to zero in the sediment results in an incorrect value of ρ at the sediment/substrate interface, with a consequent error in impedance at that boundary. This results in incorrect values of reflection loss over a much wider frequency range. We would thus expect errors in R at any grazing angle for which waves return from the lower interface. Fig. 4 compares reflection losses for the 'a = 0.519' density profile and a constant density of 1.457, at $k_0h = 30$. Differences are apparent at all grazing angles above about 16 degrees. [The two density profiles of Fig. 1 give almost identical reflection loss curves at this frequency, so no results have been plotted for the 'a = 0' profile]. We conclude from Fig. 4 that any model of sediment properties must take into account the density variation across the sediment if it is to be valid over a wide range of grazing angle. Density gradient can, however, be ignored at low grazing angles. This conclusion is consistent with that of Ref. 7 for fluid sediments.

Rutherford and Hawker's study of fluid sediments [6] concluded that, at frequencies of practical interest in underwater acoustics, it is sufficient to neglect density gradient terms in the equations of motion, and simply to apply impedance matching conditions at each interface consistent with the actual value of density at that interface. That is, they concluded that density variation effects can be modelled successfully simply by ensuring that ρ is given its correct value at each boundary. It is of interest to apply this technique for comparison with the present solution for solid sediments, but the question has not been pursued in the present work. Naturally the method would not predict reflection loss correctly at low frequency, where profile shape effects are important.

The effect of neglecting sediment shear strength is illustrated in Fig. 5, which compares reflection loss for a solid sediment and a fluid sediment. The density profile is the 'a = 0.519' profile of Fig. 1 in each case, and the frequency is $k_0h = 30$. At grazing angles above the critical angle for generation of P waves in the substrate (63.8 degrees) the two curves are indistinguishable (reflection loss being dominated by transmission of a P wave into the substrate), but the differences at smaller angles are comparable with the differences induced by neglecting the density gradient (Fig. 4). As in Fig. 4, these differences are attributed to differences in impedance at the sediment boundaries.

Finally, Fig. 6 compares a SAFARI prediction with the analytical solution, for a dimensionless frequency of 10. In this case the shear wave speed has been set to half the sound speed in both sediment and substrate. The sediment layer was represented by 20 homogeneous sublayers for the SAFARI prediction. The two curves are seen to be indistinguishable except for grazing angles in the range 35 to 50 degrees. The use of a discretised profile is therefore adequate provided a large enough number of sublayers is used. As the frequency is increased one would expect to have to use an increasingly fine discretisation to get a satisfactory result.

6. CONCLUSIONS

By assuming that the shear modulus of an inhomogeneous solid layer is small in relation to its bulk modulus, and that shear speed gradient may be neglected, it has been possible to reduce the equations of motion to a relatively simple form that clearly illustrates the conversion of P to S waves due to density gradient effects. The introduction of appropriate new variables transforms the equations to a form amenable to analytical treatment. The use of a 'sech squared' density profile and the assumption of an isovelocity sediment results in analytical solutions, which are straightforward to evaluate numerically. The existence of exact solutions is useful in enabling density profile effects to be examined without recourse to the discretised profiles employed in general numerical models such as SAFARI.

It has been demonstrated that differences in density profile shape can have a significant effect at low frequencies, where wavelengths are of the same order of magnitude as the thickness of the sediment. At higher frequencies the profile shape becomes much less important, although it is still necessary to represent the density variation within the layer so that impedance matching is done correctly at the sediment/substrate interface. This conclusion was also reached by Rutherford and Hawker [6] and Robins [7] in studies of an inhomogeneous fluid layer. For grazing angles below the critical value for generation of P waves in the substrate, neglect of density gradient can have similar magnitude effects on reflection loss as neglecting the sediment shear strength.

At frequencies of practical interest in underwater acoustics, Rutherford and Hawker have suggested that it is sufficient to characterise a sediment's density properties simply in terms of the density values at top and bottom of the layer, ensuring correct impedance matching at the sea/sediment and sediment/substrate interfaces. A future study will compare solutions obtained using that method with the present analytical solutions for a range of sediment properties, with the aim of determining the conditions in which the simple impedance matching procedure may be used.

REFERENCES

[1] J F HOOK. Separation of the vector wave equation of elasticity for certain types of inhomogeneous, isotropic media. J. Acoust. Soc. Am. 33, 302-313, 1961.

[2] F C KARAL and J B KELLER. Elastic-wave propagation in homogeneous and inhomogeneous media. J. Acoust. Soc. Am. 31, 694-705, 1959.

[3] P G RICHARDS. Weakly coupled potentials for high-frequency elastic waves in continuously stratified media. Bull. Seism. Soc. Am. 64, 1575-1588, 1974.

[4] P J VIDMAR. Ray path analysis of sediment shear wave effects on bottom reflection loss. J. Acoust. Soc. Am. 68, 639-648, 1980.

[5] M A AINSLIE. The influence of sediment rigidity on the plane-wave reflection coefficient. In 'Shear waves in marine sediments' [ed. J. Hovem et al.]. Kluwer Academic publishers, 1991.

[6] S R RUTHERFORD and K E HAWKER. Effects of density gradients on bottom reflection loss for a class of marine sediments. J. Acoust. Soc. Am. 63, 750-757, 1978.

[7] A J ROBINS. Reflection of a plane wave from a fluid layer with continuously varying density and sound speed. J. Acoust. Soc. Am. 89, 1686-1696, 1991.

[8] J LEKNER. Reflection and transmission of compressional waves: some exact results. J. Acoust. Soc. Am. 87, 2325-2331, 1990.

[9] H SCHMIDT. Seismo-acoustic fast field algorithm for range independent environments - users guide. SACLANT report SR-113, 1988.

[10] A J ROBINS. Exact solutions of the Helmholtz equation for plane wave propagation in a medium with variable density and sound speed. J. Acoust. Soc. Am. 1993.

[11] E L HAMILTON. Geoacoustic modelling of the sea floor. J. Acoust. Soc. Am. 68, 1313-1340, 1980.

[12] I TOLSTOY. Effects of density stratification on sound waves. J. Geophys. Res. 70, 6009-6015, 1965.

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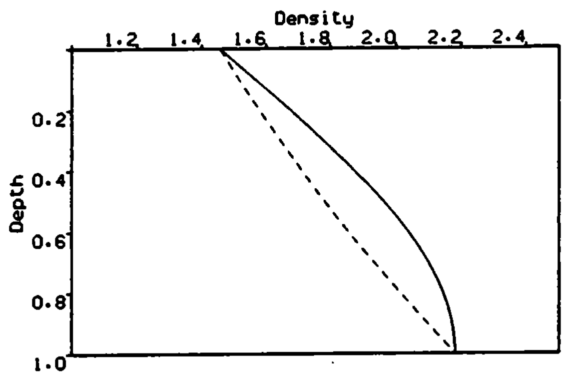

Fig.1 Density profiles
——— a=0.519; ------ a=0

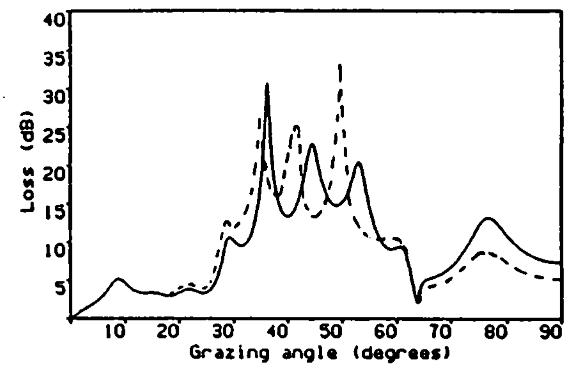

Fig.4 Reflection loss at k_0h = 30 with "a=0.519" density profile (———) and with constant density (-----)

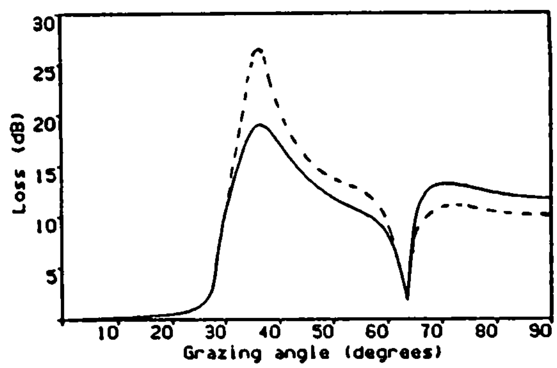

Fig.2 Reflection loss at k_0h = 2 with density profiles of Fig.1
——— a=0.519; ------ a=0

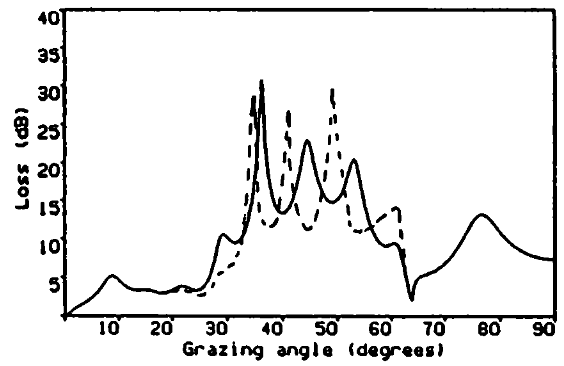

Fig.5 Reflection loss at k_0h = 30 with solid (———) and fluid (-----) sediments

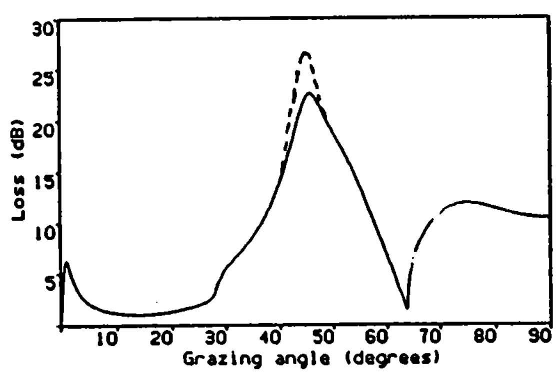

Fig.3 Reflection loss at k_0h = 8 with density profiles of Fig.1
——— a=0.519; ------ a=0

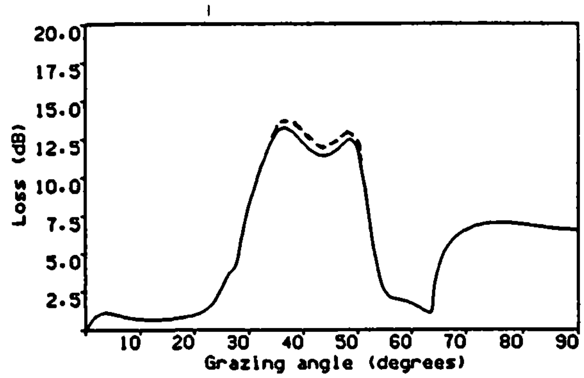

Fig.6 Comparison of analytical solution (———) with SAFARI (-----) at k_0h = 10