BRITISH ACOUSTICAL SOCIETY

71/13

"Finite Element Techniques in Structural Vibration"

24th - 25th March 71

REISSNER'S MIXED METHOD IN DYNAMICAL PROBLEMS

by

A J Sobey

Finite element methods which have been developed for the solution of vibration problems have, in the main, been based upon the principle of minimum potential energy. As such they follow the classical treatment of Ritz in synthesising the displacement field by the representation in terms of suitable base functions. The load field is obtained by first computing the strains from the displacement field and then using the elastic stress-strain relations for the material. Boundary conditions on displacement are satisfied where essential and often the additional ones are satisfied whenever it is practicable to do so.

Solutions of numerous problems have been obtained with such methods and are found to be satisfactory. There are, however, some problems where the processes stumble. Consider, for example, the vibration of a plate which has a rapid change of stress in an area where the displacements vary smoothly and in a limited way. Such a case occurs if the plate has a hole, which may well occupy a significant proportion of the plate, or where the supports exert total (or large) restraint against rotation as with a swept cantilever plate in transverse vibration. In such cases, the use of displacement methods requires careful handling.

One way out of the computational difficulty is to let the element size reduce whilst keeping unchanged the manner in which the displacement field is specified within the element. Alternatively, the displacement field can be represented by higher order terms capable of generating a load field which has the necessary rapidity of change needed to cater for the region of concentration.

It can be noted, however, that in problems of this type, that the description of the elastic system by bending stresses only may be suspect and the secondary stresses (transverse shears) could have a significant influence on the stress/displacement fields. Transverse shear effects are not easily accounted for in a displacement system and even in the relatively easier problems of static elasticity, the use of an engineer's type of bending theory based on a displacement system of given type is not prudent.

Apart from the pioneer work of Timoshenko on the influence of secondary stresses (and rotary inertia) on beam vibration, where the one dimensional character of the analysis is preserved by the use of a shear coefficient, the existence of significant secondary stresses is hard to include in a displacement system. Loads based on simple deflections which are at variance with boundary conditions would be discredited, so that the transverse shear strains computed from a Timoshenko type analysis are not reliable. The shear coefficient is an expedient used to gloss over this feature.

A second class of problems in which practical difficulties arise, but for a different reason, is that relating to systems with a compound stiffness. Consider, for example, the problem of determining the modes of a rotating beam. Part of the potential energy of the system arises from flexure against elastic restraint, in the usual way, whilst the remainder is due to flexure against the centrifugal field. For a pinned beam, with the practical dimensions and rotational speed of a helicopter blade, the elastic forces are small compared with the centrifugal ones in the lower order modes. As a result, the mode shape will be close to that of a rotating chain and in particular the fundamental flapping motion will be almost linear with radius. As a consequence, the load distribution due to the flexure (albeit small) will be very difficult to analyse by a displacement method.

Such difficulties as are common in both classes of problem cited arise from the inability of relatively uncomplicated expressions to represent a displacement field which varies only mildly from a geometrically simple form and whose higher derivatives contain the significant load information. Similar difficulties arise in static elasticity and proved obstinate to analyse either by the method of virtual forces or the method of virtual displacements.

Latterly, however, an attempt has been made to approximate both the load and displacement fields independently of each other, using Reissner's variational theorem. This states that the variation of a suitable functional can lead to the equations of equilibrium and the stress-strain relations as stationary conditions. Formally we have

$$\delta\left\{\iiint_V (\sigma_x \epsilon_x + \cdots + \tau_{xy}\gamma_{xy} + \cdots)\,dV - \iint_{S_1} (\bar{p}_x u + \cdots)\,dS\right\} = 0$$

where $\sigma_x, \ldots$ are direct stresses, $\tau_{xy} \ldots$, shear stresses, $\epsilon_x \ldots$, direct strains, $\gamma_{xy} \ldots$, shear strains, $\bar{p}_x \ldots$, prescribed surface stresses over part S_1 of boundary, u,.. displacements and V enclosed volume.

Several novel and accurate approximate solutions to problems not characterised by the domination of the elastic behaviour by primary stresses have followed from Reissner's theorem. By way of illustrations, Reissner has shown how the effect of transverse shear stresses may be taken with account in the flexure of plates and an interesting study of the non-uniform torsion of cylindrical rods. In the latter case, the flexibility of solution when both stress and displacement fields may be varied is seen to striking advantage.

If, in the Reissner variational theorem, we allow continuous variation of the load field and impose no restriction whatsoever on its behaviour, then the stationary conditions generate the classical displacement analysis with the governing relations normally obtained by Newtonian methods. Such indeterminacy in the stresses fails to exploit the potential of the Reissner method and it is generally advantageous in seeking approximate solutions to represent stress and displacement fields by suitable functions which do not violate too severely the stress-strain relations. In this way, economy is combined with reasonable accuracy.

In dynamical problems, we may use either the quasi-static concept in which the structure is loaded inertially, or we may invoke Hamilton's principle, using the Reissner integral as potential energy function. For conservative systems, the two procedures will reduce to the same resulting form.

In finite element applications we may use the variational principle to determine the load and displacement fields by prescribing the load and displacement variation within an element and using the variational method to find the interface values of the unknown load and displacement parameters.

A sketch follows of the mixed method applied to the problem of the determination of the normal modes of a rotating beam. If w is the deflection of a blade element at radius r parallel to the axis of rotation, m the blade mass per unit span, EI the flapping stiffness of the blade T the tension in the blade at radius r, and Ω the angular velocity of the blade,

then $$T = \Omega^2 \int_r^{R} m\, r\, dr$$

It is easily shown by Newtonian methods that the equation of motion when the blade is constrained to flap only is

$$m\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial r}\left(T\frac{\partial w}{\partial r}\right) + \frac{\partial^2}{\partial r^2}\left(EI\frac{\partial^2 w}{\partial r^2}\right) = 0$$

with boundary conditions $w(r_0,t) = \dfrac{\partial^2 w(r_0,t)}{\partial r^2} = \dfrac{\partial^2 w(R,t)}{\partial r^2} = \dfrac{\partial^3 w(R,t)}{\partial r^3} = 0$

for a blade hinged at $r = r_0$ and having the tip $r = R$ free of stress.

Solutions of this system, which is a classical eigenvalue problem can be obtained by a variety of approximate processes, of which one of the most suitable is that of Wadsworth and Wilde.

A finite element approach based on Reissner's principle follows. If the blade deformation is governed by primary bending stresses only,

$$\sigma_r = M\eta/EI$$

where η is the distance of a blade fibre from the neutral axis in bending and M is the resultant bending moment in the blade. The strains are

$$\epsilon_r = \frac{\partial^2 w}{\partial r^2}\eta$$

and in this case we do not presume, ab initio, that $M = EI\,\partial^2 w/\partial r^2$ M and w are to be thought of as independent functions which will be tied together through the stationary properties of the appropriate functional.

By integration using the Reissner integral across the beam section, the modified Lagrangian in this case is found to be

$$L' = \frac{1}{2}\int_{r_0}^{R}\left\{ m\left(\frac{\partial w}{\partial t}\right)^2 - T\left(\frac{\partial w}{\partial r}\right)^2 + \frac{M^2}{EI} - 2M\frac{\partial^2 w}{\partial r^2}\right\} dr$$

so that Hamilton's principle gives

$$\delta\int_{t_1}^{t_2} L'\, dt = 0$$

Allowing M and w to be arbitrary functions of space-time, subject only to the essential boundary conditions we find

$$\int_{t_1}^{t_2}\int_{r_0}^{R}\left(\left\{-m\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial r}\left(T\frac{\partial w}{\partial r}\right) - \frac{\partial^2 M}{\partial r^2}\right\}\delta w + \left\{\frac{M}{EI} - \frac{\partial^2 w}{\partial r^2}\right\}\delta M\right) dr\, dt$$
$$+ \int_{r_0}^{R}\left[m\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2} dr + \int_{t_1}^{t_2}\left[-T\frac{\partial w}{\partial r}\delta w - M\frac{\partial}{\partial r}(\delta w) + \frac{\partial M}{\partial r}\delta w\right]_{r_0}^{R} dt = 0$$

Since δw and δM are arbitrary functions of r,t within the limits, we require both

$$-m\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial r}\left(T\frac{\partial w}{\partial r}\right) - \frac{\partial^2 M}{\partial r^2} = 0$$

and $$M/EI - \frac{\partial^2 w}{\partial r^2} = 0$$

Suppose, now, we restrict M, w to be harmonic functions of time

$$M(r,t) = \sin\omega t\, M(r) \qquad w(r,t) = \sin\omega t\, w(r).$$

The single integrals in Hamilton's principle will all vanish if in addition to boundary conditions on w, M we take $t_1 = 0$, $t_2 = 2\pi$.

The integral is then reduced to one in the radial coordinate only and the new form of the variational equation is

$$\delta \int_{r_0}^{R} \left\{ m\omega^2 w^2 - T\left(\frac{dw}{dr}\right)^2 + \frac{M^2}{EI} - 2M\frac{d^2w}{dr^2} \right\} dr = 0$$

As it stands, this variational equation could be used as the basis of a (single) finite element method, but whilst the process might be made to work, it is pointless to try to do so when partitioning the structure into a number of discrete elements affords more profitable use of computer space.

Divide the beam into N arbitrary elements. The load field is specified completely by the bending moment distribution. The displacements and the bending moments can be defined in any convenient way subject to certain continuity conditions at the interfaces. If w and M and their first derivatives (slope and shear force) are all made continuous (a condition severer than is needed to ensure a proper variational condition) then the simplest interpolative rule within an element is to fit third order (cubic) polynomials in r for M and w.

For the hinged beam, the variational process leads to a matrix equation of order 4N which splits into $Pq_1 - Qq_2 = 0$ and $Q'q_1 + (R - \omega^2 S)q_2 = 0$

where $$q_1 = \{M_1, M_2, \cdots, M_n, M_0', \cdots, M_{N+1}'\}$$

and $$q_2 = \{w_1, w_2, \cdots, w_n, w_0', w_1', \cdots, w_N'\}$$

where M_r and w_r are the interface values of M and w between the rth and $(r + 1)$th element, and primes denote radial derivatives.

The system properly of order 4N condenses into one of order 2N + 1 by elimination of q_1 giving the homogeneous system for q_2 of the form

$$[R + Q'P^{-1}Q - \omega^2 S]\, q_2 = 0$$

Here P is a symmetric positive definite matrix derived from beam stiffness terms, Q is a general matrix derived from the (Md^2w/dr^2) variation, R derives from centrifugal stiffness and is symmetric positive definite and S is the inertia matrix.

The reduced equation for q_2 has positive definite matrices in $(R + Q'P^{-1}Q)$ and S and so leads to 2N + 1 real values of ω, as is to be expected for a conservative system.

Numerical illustrations will be presented at the lecture.

The method is seen at its most potent in situations where the conditions are more complex than the one dimensional one used for illustration, and all of the introductory examples would be solved readily by the use of independent load and displacement fields.

The method does, however, possess one disadvantage which compares unfavourably with the Ritz process. The stationary condition arising from Hamilton's principle using the Reissner representation of elastic energy is not extreme and so the error in the eigenvalues is of indeterminate sign. In the Ritz process we always overestimate the frequencies and are assured of the sign, if not the magnitude of the error, in the mixed method we are sure of neither.