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NUMERICALLY CALCULATED INTRINSIC MODE FIELDS.

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1. INTRODUCTION

The study of the propagation of low frequency acoustic fields through 2-dimensional ocean environments – wherein the height of the ocean channel, overlying a fluid bottom, decreases linearly with range – has generated significant interest in recent years [1,2]. The pressure field in this variable depth ocean waveguide cannot be determined by applying separation of variables or transform techniques, because of the inherent weak range dependence. This weak non-separability does permit approximate analytical methods, by assuming that on each local cross-section the field consists of the transverse modes of the local cross-section. Two methods based on this assumption are coupled modes [3] and adiabatic modes [4].

Solutions of weak range dependent problems can also be generated by fast and stable numerical routines. Development of this type of routine is through approximation of the elliptic equation by a parabolic equation. The analysis is discussed extensively by Tappert [5], the essential feature of the work being that the boundary value problem of interest is approximated by an initial value problem, allowing stepwise integration throughout the range dependent structure. The Parabolic Equation Method (PEM) [5] and the Beam Propagation Method (BPM) [6] are two of the marching algorithms resulting from this approximation.

The wedge shaped ocean of Jensen and Kuperman [7] is one of the simplest non-separable structures, and comparisons between various numerical solutions are informative, provided a benchmark solution is available, as all the above methods are approximate even in the limit of zero propagation step.

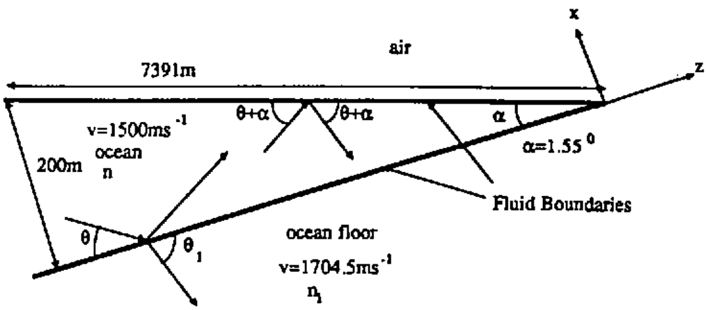
Recent developments using spectral synthesis have obtained global solutions to the wedge shaped ocean which adapt intrinsically to the local wedge environment. These 'Intrinsic Modes' [8] are global spectral objects which are exact solutions of the wave equation except at the wedge apex. Consequently the Intrinsic Mode field is a benchmark solution for the wedge ocean environment and analysis of the performance of other schemes is possible. Although the structure is large scale, the transfer of guided modes into radiation modes will occur over a few wavelengths regardless of size of the wedge angle. The determination of this transition region (mode cut-off), by approximate algorithms, is a useful mesoscale phenomenon to examine.

Firstly a brief description of the derivation of the different approximate solutions and the exact Intrinsic Mode solution for the wedge shaped ocean will be given.

2. APPROXIMATE METHODS

Attention is confined to modal cut-off in the two dimensional wedge shaped ocean discussed by Jensen and Kuperman [7] which is depicted in Figure 1. It is apparent from the geometry of the structure that the field propagating upslope has weak range dependence. The essential feature of the approximate solutions derived below is the exploitation of this

Figure 1: The Jensen -Kuperman Wedge Environment



weak range dependence.

The Adiabatic Modal field in this two dimensional geometry is constructed as a sum of local transverse modes of the local transverse cross-section with a global phase propagation incorporated. This type of field can be represented in the form

$$U_{am}(z,x) = \sum_q \Phi_q(\epsilon z,x)e^{i\Psi_q(z)} \qquad (2.1)$$

with ϵz representing weak dependence on range of the transverse mode $\Phi_q(\epsilon z,x)$. $\Psi_q(z)$ is the global phase of each mode at the local cross-section defined by z. Substituting (2.1) into the 2-dimensional scalar Helmholtz equation yields two 1-dimensional equations; one confirms the transverse modal nature of $\Phi_q(\epsilon z,x)$ and the other determines $\Psi_q(z)$. Weak non-separability implies the coupling between modes may be neglected, the energy in each mode remains constant and the second derivative of the phase, $\Psi_q(z)$, with respect to the longitudinal parameter z may be neglected. For small wedge angles of the Jensen-Kuperman model ocean the Adiabatic Mode field is

$$U_{am}(z,x) = \left[\frac{\tau_q}{\beta_q(1+\tau_q h(\epsilon z))}\right]^{-\frac{1}{2}} \exp i\int_{z_0}^{z}\beta(z')dz' \begin{cases} \sin\left[\gamma_q(h(\epsilon z)-x)\right] & 0<x<h(\epsilon z) \\ \sin(\gamma_q h(\epsilon z))e^{\tau_q x} & x<0 \end{cases} \qquad (2.2)$$

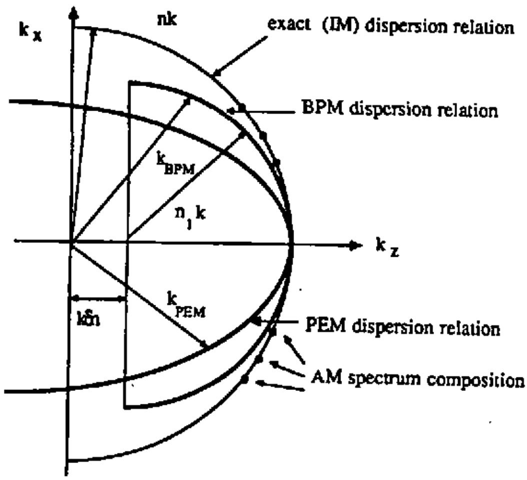
The modal parameters are defined using the refractive index of the structure (determined from the velocity of the pressure field) and are

$$\gamma_q^2 = n^2k^2 - \beta_q^2 \qquad \tau_q^2 = \beta_q^2 - n_1^2k^2 \qquad -\gamma_q\cot(\gamma_q h) = \tau_q \qquad (2.3)$$

The AM field produces a non-physical phenomenon at the modal cut-off, the explanation of which is given later. This particular AM solution is specific to the cross-section environment encountered in the ocean wedge problem as each adiabatic mode is determined entirely by the eigenvalue equation (2.3) at each local cross-section.

The PEM and the BPM are more general approximate solutions of the scalar Helmholtz equation. Provided the variation in sound speeds with respect to a background speed is small,

Figure 2: Wave vector loci of numerical models.



the field is paraxial, and back reflection processes are negligble, then in principle application of the PEM and BPM is legitimate. It will be demonstrated that although application may appear legitimate, in some instances, discrepancies between methods may occur. The derivation of the two dimensional PEM starts from the premise that the field is predominantly travelling in the z direction. i.e.

$$U_{pem}(z,x) = G(\epsilon z,x)e^{in_1kz} \tag{2.4}$$

Consquently the PEM field can be viewed as a plane wave propagating in the z direction with a transverse envelope function which changes slowly with range. By substituting (2.4) into the acoustic Helmholtz equation, and neglecting the second derivative of $G(\epsilon z,x)$ with respect to range, the envelope function $G(\epsilon z,x)$ must satisfy the equation

$$\left[\frac{\partial^2}{\partial x^2} + 2in_1k\frac{\partial}{\partial z} + k^2(n^2(\epsilon z,x)-n_1^2)\right]G(\epsilon z,x) = 0 \tag{2.5}$$

The generation of the envelope function $G(\epsilon z,x)$ for given initial conditions on a fixed transverse cross-section is achieved by application of Fourier transform methods. At each cross-section the spectrum is obtained by application of an FFT. Each plane wave constituent is then propagated to the next transverse cross-section by addition of a k_z phase term obtained from the dispersion relation of (2.5), given in (2.6) and shown in Figure 2.

$$k_x^2 + 2n_1kk_z = (n^2(\epsilon z,x) - n_1^2)k^2 \tag{2.6}$$

The PEM field is obtained by the inverse Fourier transformation of this propagated spectrum, via the FFT. PEM is a marching algorithm method because the field is evaluated at each transverse cross-section as the observation point moves forward through the structure.

The BPM field is of the marching algorithm type, but in this case the sound velocity at each point in the structure is treated as a small pertubation on a chosen background velocity. In refractive index terms this is represented as

$$n(z,x) = n_1 + \delta n(\epsilon z,x) \tag{2.7}$$

At a particular cross-section the field is Fourier transformed and propagated to the next cross-section as though the medium had a constant refractive index n_1. At this cross-section a phase term is added which then corresponds to propagation through the pertubation δn (the thin lens approximation [6]). These two phase additions can be calculated from the dispersion relation of the BPM wave equation. The wave equation and its dispersion relation are given in equation (2.8) and (2.9) repsectively, with the latter depicted in Figure 2.

$$\left[\nabla^2 - 2ik\delta n(x)\frac{\partial}{\partial z} + k^2(n_1^2-\delta n^2(x)) \right]U_{bpm}(z,x) = 0 \tag{2.8}$$

$$k_x^2 + \left[k_z - k\delta n(x)\right]^2 = n_1^2k^2 \tag{2.9}$$

3. THE INTRINSIC MODE

The previous methods generate a solution which under certain constraints is a good approximation to the solution of the elliptic wave equation. However, even in the limit of zero propagation step the approximate solutions generated by AM, PEM, and BPM will not be exact solutions of the scalar Helmholtz equation. It is desirable to test and evaluate the performance of approximate algorithms with respect to an exact benchmark solution. The Intrinsic Mode field [8], synthesised in the spectral domain, is such a benchmark solution. Construction of the Intrinsic Mode is based on superposition of known solutions to the wave equation (planewaves) in order to satisfy the boundary conditions. The strategy is as follows:– the field U(z,x) in a medium n, with a wave number k, can be represented as the sum of upward and downward planewaves from a chosen datum. The upward and downward field can be represented in the form

$$U^{\pm}(z,x) = \int_{C^{\pm}} \bar{U}^{\pm}(\theta)e^{ink(z\cos\theta\pm x\sin\theta)}d\theta \tag{3.1}$$

with + and – signs denoting upward and downward fields respectively, and θ the angle of the plane wave with respect to the lower boundary shown in Figure 2. The contours $C^{\pm}$ range over all propagation directions and fully account for inhomogeneous wave propagation a full derivation is given in [8]; but it suffices to treat the integral as ranging over an interval $(0,\Pi/2)$ (i.e. forward propagation). The evaluation of $U^{\pm}(\theta)$ will then determine, via the transformation of (3.1), the field in the wedge geometry. The upward and downward fields must satisfy the boundary conditions. Consider a plane wave travelling towards the upper boundary at angle θ to the lower boundary, which has an amplitude $U^{+}(\theta)$. After reflection in the upper boundary the plane wave is travelling downwards at an angle $\theta+2\alpha$ to the lower boundary and as a consequence has an amplitude $U^{-}(\theta+2\alpha)$. Thus for the upward and downward fields to be consistent the amplitude of the downward propagating field must equal the upward propagating field multiplied by a reflection coefficient due to the interaction with the upper boundary. i.e.

$$U^{+}(\theta)e^{i\Phi_u(\theta+\alpha)} = U^{-}(\theta+2\alpha) \qquad (3.2)$$

At the bottom boundary the relation between the upward and downward fields is

$$U^{+}(\theta) = U(\theta)e^{i\Phi_l(\theta)} \qquad (3.3)$$

with Φ_u and Φ_l being the phase reflection coefficients at the upper and lower boundaries respectively. These relations are shown diagramatically in Figure 2. In the Jensen–Kuperman ocean the phase of the reflection coefficients are

$$\Phi_u(\theta) = \Pi, \qquad \Phi_l(\theta) = 2\tan^{-1}\left[\frac{in_1\rho\sin\theta_1}{n\rho_1\sin\theta}\right], \quad n\cos\theta = n_1\cos\theta_1 \qquad (3.4)$$

From (3.2) and (3.3) a recurrence relation for the spectrum can be found which can be solved exactly, upto 2α periodic functions of θ, via the Euler–Maclaurin formula [8]. The field in the lower medium is obtained by application of a transmission coefficient to the downward spectrum and the correct planewave propagation. The Intrinsic Mode throughout the wedge is then

$$W_q(\underline{x}) = \begin{cases} \sum_{+-} \int_{C^\pm} e^{iS_q^\pm(\theta)} e^{-inkr\cos(\theta\pm\chi)} d\theta, & \underline{x} \in X \\ \int_{C^-} e^{iS_q^-(\theta)}\left[1 + e^{i\Phi_l(\theta)}\right] e^{-in_1kr\cos(\theta-\chi)} d\theta, & \underline{x} \in X_1 \end{cases} \qquad (3.5)$$

with,

$$S_q^-(\theta+2\alpha) = \frac{\Phi^-(\theta)}{2} + \frac{1}{2\alpha}\int_{\theta_c}^{\theta} \Phi^-(s)ds - \frac{q\Pi\theta}{\alpha} + E_b^-(\theta,\theta_q) \qquad (3.5)$$

and,

$$S_q^+(\theta) - S_q^-(\theta) = \Phi_l(\theta)$$

$$\Phi^-(\theta) = \Phi_l(\theta) + \Phi_u(\theta+\alpha) \qquad (3.5)$$

4. COMPARISON OF NUMERICAL SCHEMES

The q=2 Intrinsic Mode for the wedge structure is indicative of how the guided field varies slowly on a wavelength scale until it reaches a critical point, cut–off, wherein a rapid change from the guided regime to the radiative regime is encountered. It is this transition region and its subsequent modelling by algorithms discussed in sections 2 and 3 which is of interest here.

The J−K ocean of Figure 1 will support 3 dominant Intrinsic Mode fields. In the following coparison each of these fields is to be launched upslope from a chosen cross−section (ocean height = 200m) and propagated throughout the wedge environment using the four different methods. The central difficulty is how to present the data. The use of contour plots is limited because the resolution is insufficient to discern appreciable differences in the solutions. If, however, the amplitudes of the fields along the interface between the ocean and the ocean floor are compared, useful conclusions may be drawn. Figure 3 show the fields along this interface as a function of local guide height for the four methods with the 3 different Intrinsic Modes launched from the cross−section with an ocean depth of 200m.

Modal cut−off is defined as the range at which the propagation constant of the local normal mode, β_q, equals the propagation constant of a plane wave in the lower medium. For a given wavelength and refractive index change the cut−off of a mode is dependent only on the height of the guiding channel. This cut−off height, h_c, is

$$h_c = \frac{(q - \frac{1}{2})2\Pi}{k(n^2 - n_1{}^2)^{\frac{1}{2}}} \qquad (4.6)$$

The denominator is the transverse wave number in the guiding layer and calculation of cut−off, through the evaluation of the transverse wave number, by the approximate schemes is a measure of their performance. The AM, PEM and the Intrinic Mode have identical transverse wave numbers at the appropriate cut−off point. The BPM calculates the transverse wave number at cut−off to be

$$k_{BPM} = k(\delta n(2n_1 - \delta n))^{\frac{1}{2}} \qquad (4.7)$$

Table 1 below shows the discrepancy between cut−off heights of the BPM and other schemes. The cut−off heights observed in Figure 3 correspond to those of Table 1. As the AM propagates upslope the energy density increases as the height decreases for no energy is transfered to any other mode.

Table 1 Local Cut−off heights for the J–K ocean.

Mode Number	BPM cut-off height	PEM + IM cut-off heights
1	33.81m	31.57m
2	101.42m	94.72m
3	169.03m	157.87m

At cut−off the AM has a constant evanescent field, which can only contian finite power if the constant is zero. Consequently at cut−off the AM field is zero, a non−physical occurence.

INTRINSIC MODES

The PEM and BPM are continuous through cut-off and demonstrate cut-off in their predicted positions. The PEM and BPM have oscillations which can be explained by an intrinsic mode analysis. The PEM (BPM) satisfies particular paraxial equations, which will possess particular PEM (BPM) intrinsic modes. Thus when the structure is excited with a pure Intrinsic Mode all the PEM (BPM) intrinsic modes are excited. PEM (BPM) intrinsic modes will have very similar properties to pure Intrinsic Modes and their cut-off points will be as predicted in table 1. These cut-off points are clearly observed in Figure 3(c).

Figure 3(a)

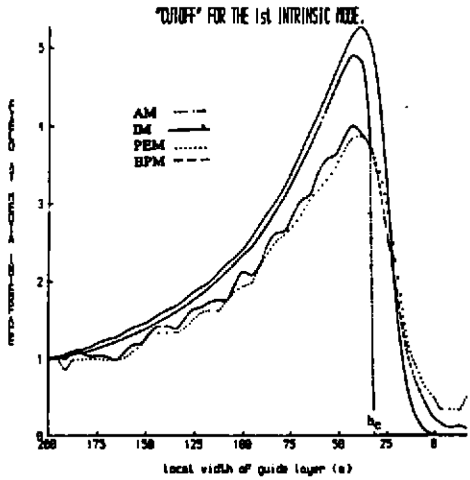


Figure 3(b)

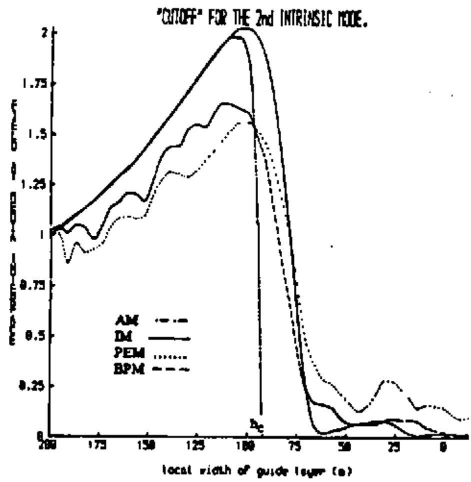
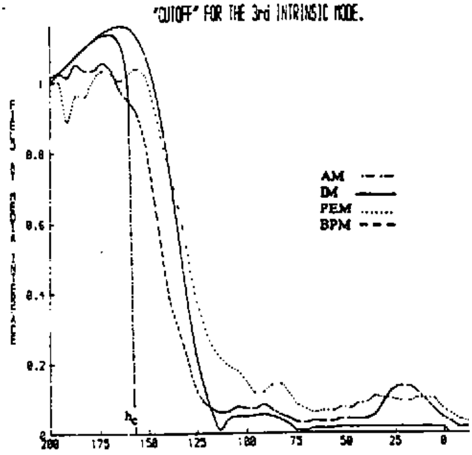


Figure 3(c)



5. CONCLUSIONS

The Intrinsic Mode in a wedge shaped ocean waveguide is compared numerically with PEM, BPM and AM calculations, along the interface between the water and the sea bottom. The phenomena of mode cut-off and radiation into the bottom are clearly visible, in the progressive increase, then sudden decrease of the field values with distance along the interface. Although the qualitative behaviour of the fields is the same for all methods, there are discernable numerical differences between them. In particular the IM agrees closely with AM in the region where the AM is guided, but of course this agreement does not survive the cut-off transition. Oscillations in the PEM and the BPM calculations do not appear in the IM field, because the initial condition used to initiate the calculations exite only <u>one</u> true IM of the Helmholtz equation, but excites several intrinsic modes of the PEM or BPM, which mutually interfere. The cut-off of these PEM and BPM intrinsic modes can clearly be seen in Figure 4(c). The IM computation is much more efficient than either PEM or BPM for this particular choice of field points, because the field values along the bottom interface can be computed by one application of the FFT to (3.5). Also the IM is essentially an <u>exact</u> solution of the wedge boundary-value problem, though it is approximated to make its computation more tractable. We wish to acknowledge the contribution of Dr. J. Gribble for his analysis of the BPM, and many useful discussions.

7. REFERENCES

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