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EFFECT OF ADDED MASS ON A VIBRATING RING

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INTRODUCTION

This paper outlines a theoretical method for the computation of natural frequencies and mode shapes of a uniform circular ring with any system of discrete masses attached to the surface. Simplification is affected by neglecting shear force and rotary inertia effects and only flexural vibrations in the plane of the ring are considered.

A uniform ring is considered to comprise a finite number of lumped masses connected by massless members of equal stiffness to the segments they represent. Such a simplification may subsequently be modified to obtain considerable improvement in accuracy, [RAUSCHER(1)] .

Since the ring is unconstrained, it is necessary to determine the singular stiffness matrix and then the mass matrix for the ring is modified to take account of the added masses. The eigenproperty solution of the resulting dynamic matrix yields natural frequencies and modal patterns.

Comparisons with experimental results and, for a simple case, with Classical theory [TIMOSHENKO(2)] , indicate good accuracy for the method.

NOTATION

A	Area of cross-section	x,y	2.D Cartesian co-ordinates
E	Young's modulus	θ	Rotational co-ordinate
I	Second moment of area	H,V	Forces in x- and y-directions respectively
N	Number of elements	M	Bending moment
R	Mean radius	$[K]$	Stiffness matrix
k	Radius of gyration	$[M]$	Mass matrix
β	$= R^2/k^2$	$[\delta]$	Displacement matrix
ϕ	Angle subtended by consecutive elements	Subscripts: A,B,C	Refer to locations (see Fig. 1)
w	Circular frequency		

STIFFNESS MATRIX

Method of Derivation

Stiffness matrix coefficients may be determined conveniently by the Displacement Method [3]. Each elemental mass must be allowed three degrees of freedom, representing tangential, radial and rotational displacements. Thus, for a typical element C (Fig.1), the required coefficients are represented by the forces and bending moments at A,B and C for each of three displacement cases, namely:-

CASE 1 $\quad x_c = 1,\ x_{A,B} = 0,\ y_{A,B,C} = 0,\ \theta_{A,B,C} = 0$ .....(i)

CASE 2 $\quad x_{A,B,C} = 0,\ y_c = 0,\ y_{A,B} = 0,\ \theta_{A,B,C} = 0$ .....(ii)

CASE 3 $\quad x_{A,B,C} = 0,\ y_{A,B,C} = 0,\ \theta_c = 1,\ \theta_{A,B} = 0$ .....(iii)

For a uniform ring, the stiffness of each element is identical and the complete stiffness matrix may be derived by computing six forces and three bending moments for each of the above three cases of a built-in circular segmental-arch.

Segmental-Arc Cantilever Theory

Because of symmetry of arch ACB (Fig.1), a segmental-arc cantilever may be isolated as shown in Fig.2; deflections are given in terms of the general force system by existing theory [4] :-

$$\theta = \frac{R^2}{EI}\left\{\frac{M_c\phi}{R} - H_c(\phi - \sin\phi) - V_c(1-\cos\phi)\right\} \quad .....(1)$$

$$x = -\theta R + \frac{\beta R}{4AE}\left\{\frac{4M_c\sin\phi}{R} + H_c(2\phi + \sin 2\phi - 4\sin\phi) - V_c(1-\cos 2\phi)\right\} + \frac{R}{4AE}\left\{H_c(2\phi + \sin 2\phi) - V_c(1-\cos 2\phi)\right\} \quad .....(2)$$

$$y = \frac{\beta R}{4AE}\left\{\frac{4M_c}{R}(\cos\phi - 1) + 2H_c(1-\cos\phi)^2 + V_c(2\phi - \sin 2\phi)\right\} - \frac{R}{4AE}\left\{H_c(1-\cos 2\phi) - V_c(2\phi - \sin 2\phi)\right\} \quad .....(3)$$

Equations (1)-(3) are solved for Hc, Vc, Mc for each of the displacement conditions (i), (ii), (iii) above. Forces and moments at A are determined by the equations of static equilibrium. The right-hand-side of arch ACB may be considered in a similar manner to the left-hand-side. After combining the two sides and resolving forces in the radial and tangential directions, the nine influence coefficients for each of Cases (1)-(3) are determined.

By ignoring rotary inertia, the stiffness matrix is reduced from order (3N x 3N) to order (2N x 2N).

MASS MATRIX

This is a diagonal matrix of order (2N x 2N) whose coefficients are the mass values of ring elements, including any attached masses, written out twice in order.

THE EIGENVALUE PROBLEM

The mass and reduced stiffness matrices for the complete assembly are now inserted in the standard eignevalue equation [5]:-

$$\left([K] - \omega^2[M]\right)\{\delta\} \quad ......(4)$$

Solving this for eigenvalues and eigenvectors yields the square of circular vibration frequency and relative displacement amplitudes respectively.

Since $[K]$ is symmetric and $[M]$ is both symmetric and positive definite,use is made of Choleski decomposition to employ Jacobi rotation solution of a reduced Dynamic matrix which is symmetric. Standard ICL library subroutines enable simultaneous solution of all the eigenproperties relating to the original system. Computation times vary from 40 seconds, using 16 elements, to about $1\frac{1}{2}$ minutes, using 24 elements, on an ICL 1904A, 128K computing system.

EXPERIMENTAL WORK

A 15.5 inch diameter steel ring, of 0.25 in (radial) x 0.30 in rectangular section and of mass 1.035 lb., was supported horizontally by three vertical slender needles at equal angular spacing. Point excitation in the radial sense (and also one of the added masses!) was provided by an electro-dynamic exciter and vibration measurement was by means of a proximity electro-magnetic pick-up, with Cathode Ray oscilloscope instrumentation. Natural frequencies and modal patterns were determined for the unloaded ring and the procedure repeated for various added mass configurations.

SAMPLE RESULTS

1. Uniform Ring (no added mass)

Natural Frequencies:-

MODE	CLASSICAL THEORY	16 ELEMENTS	24 ELEMENTS	* EXPERIMENT
1	104.0	104.0	104.0	100
2	294.3	293.9	294.2	278
3	564.2	561.0	563.7	532
4	912.5	894.5	910.0	855
5	1339	1262	1330	1225

* Note: Experimental results are generally low due to the added mass effect of the exciter moving parts.

Mode Shapes:- In all cases tabulated above, the first two modal patterns showed good correlation with Fig.3(a) and (b) respectively.

2. Uniform Ring with Mass Addition as shown below (Fig. 4)

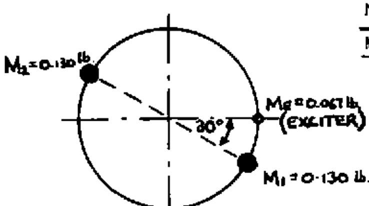


Fig. 4. Added mass system for Example 2

Natural Frequencies:-

MODE	24 ELEMENTS	EXPERIMENT
1	86.2	86.5
2	275.4	276.5
3	540.2	523
4	767.3	825
5	1142	1170

Mode Shapes:- The 4 and 6 noded Modal patterns obtained both experimentally and theoretically for the ring with two added masses (+ exciter mass, Fig. 4) closely resemble the theoretical modal patterns for a uniform ring (Fig.3), obtained from classical theory. The reference axes are, however, positively located by the added masses as shown in Fig. 4.

DIAGRAMS

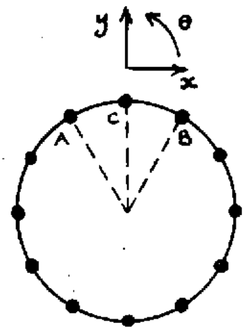


Fig. 1. MATHEMATICAL MODEL

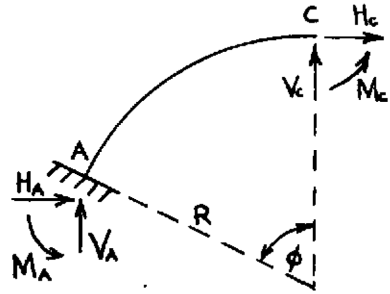
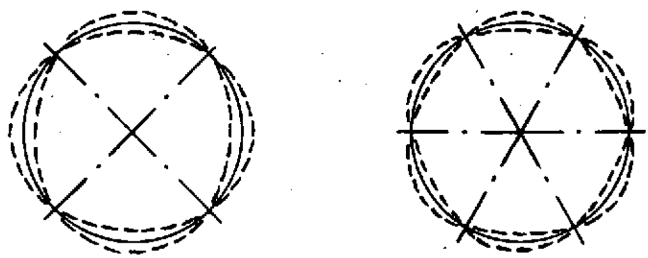


Fig. 2. SEGMENTAL-ARC CANTILEVER



(a) First Mode

(b) Second Mode

Fig. 3. FIRST TWO NATURAL VIBRATION MODES OF A UNIFORM CIRCULAR RING

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