# ACCURATE SOURCE LOCALIZATION – THEORETICAL AND EXPERIMENTAL RESULTS

BJÖRN OTTERSTEN AND MATS VIBERG

*Dept. of EE, Linköping University, S-581 83 Linköping, Sweden.*

## 1 Introduction

High-resolution signal parameter estimation is a problem of significance in many signal processing applications. These include emitter location estimation, jammer suppression, system identification, and time series analysis. Sensor arrays play an important role in areas such as sonar, radar, and electronic surveillance, where the objective is to detect and estimate incoming signals. Signal parameters of interest can include azimuth and elevation angle, temporal frequency, etc. The availability of accurate and less expensive analog to digital converters allows the design of arrays where each sensor output is digitized individually. This greatly expands the signal processing possibilities for the array data as compared to analog processing methods.

Much of the recent work in array signal processing has focussed on methods for high-resolution location estimation. When the emitter signals are generated by spatially close sources, conventional *beamforming* methods fail to separate the angles-of-arrival (AOAs). Several different methods have been proposed for estimating closely spaced AOAs, [1, 2, 3, 4, 5, 6]. Herein, a unified *subspace fitting* framework is presented, establishing algebraic and asymptotic connections between a large class of these algorithms. Preliminary results from processing experimental data from the Baltic sea are also included. A significant increase in estimation accuracy can be noted for the subspace based estimation procedure compared to traditional beamforming. We conclude that high-resolution model based estimation procedures can provide enhanced performance even under non-ideal conditions.

## 2 Problem Formulation

Consider an array of $m$ sensors at which $d$ narrow band plane waves are arriving. Let a collection of parameters be associated with each emitter signal. These may include bearing, elevation, range, polarization, carrier frequency, etc., and will be referred to as signal parameters. Let the parameter vector $\theta_i$ denote the collection of parameters associated with the $i^{th}$ signal, $s_i(t)$. The response of the $k^{th}$ sensor and the time delay of propagation for the $i^{th}$ signal at the $k^{th}$ sensor are denoted by, $\rho_k(\theta_i)$ and $\tau_k(\theta_i)$ respectively. The following *parametrized* data model is then obtained for the $k^{th}$ sensor output

$$x_k(t) = \sum_{i=1}^{d} \rho_k(\theta_i) e^{-j\tau_k(\theta_i)} s_i(t) + n_k(t) = \sum_{i=1}^{d} a_k(\theta_i) s_i(t) + n_k(t) \; , \qquad (1)$$

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where $a_k = \rho_k e^{-j\gamma_k}$ and where $n_k(t)$ represents measurement noise. The sensor array is composed of $m$ elements, and the individual sensor outputs are collected in an output vector

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^{d} \mathbf{a}(\theta_i)s_i(t) + \mathbf{n}(t) \\ &= [\mathbf{a}(\theta_1)\ldots\mathbf{a}(\theta_d)][s_1(t)\ldots s_d(t)]^T + \mathbf{n}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t)\ . \end{aligned} \tag{2}$$

The true signal parameters are denoted $\theta^T = [\theta_1^T \ldots \theta_d^T]$ .

A crucial assumption in the techniques to be discussed is the knowledge of the array parametrization. The array response to a wavefront with parameters $\theta_i$ is denoted $\mathbf{a}(\theta_i)$. This *array response vector* is an element of a complex $m$-dimensional vector space. As the parameters, $\theta_i$, vary over the parameter range of interest, the array response vector defines a manifold in the space. This array manifold is assumed known as a function of $\theta_i$.

The emitter signals and the noise are assumed to be independent random processes with covariance matrices

$$\mathrm{E}\{\mathbf{s}(t)\mathbf{s}^*(t)\} = \mathbf{S}\ , \qquad \mathrm{E}\{\mathbf{n}(t)\mathbf{n}^*(t)\} = \sigma^2\,\mathbf{I}\ . \tag{3}$$

In addition, the asymptotic analysis briefly mentioned later requires the noise vector to be temporally white and complex Gaussian distributed. The spatial covariance of the array output takes the form

$$\mathbf{R} = \mathrm{E}\{\mathbf{x}(t)\mathbf{x}^*(t)\} = \mathbf{A}(\theta)\mathbf{S}\mathbf{A}^*(\theta) + \sigma^2\,\mathbf{I}\ . \tag{4}$$

If the emitter signal waveforms are non-coherent, the signal covariance matrix, $\mathbf{S}$, has full rank. However, in many applications specular multipath is common and $\mathbf{S}$ may be ill-conditioned or even rank deficient. In general, let the rank of the $d \times d$ signal covariance matrix be $d'$.

Assume that the sensor outputs are measured simultaneously at $N$ time instants, $t_1, \ldots, t_N$. Each vector observation is called a *snapshot* of the array output. The *data matrix* is the collection of array snapshots and is modeled by

$$\mathbf{X}_N = [\mathbf{x}(t_1)\ldots\mathbf{x}(t_N)] = \mathbf{A}(\theta)\mathbf{S}_N + \mathbf{N}_N\ . \tag{5}$$

The sample covariance matrix is defined as

$$\hat{\mathbf{R}} = \frac{1}{N}\mathbf{X}_N\mathbf{X}_N^* = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}(t_i)\mathbf{x}^*(t_i)\ . \tag{6}$$

Given the observations $\mathbf{X}_N$ and a model for the array response $\mathbf{a}(\theta_i)$, the main objective for our purposes is to estimate the signal parameter vector $\theta$. The problem of detecting the number of signals, $d$, and/or to separate the individual signal waveforms, $s_i(t)$, is not discussed.

## 3 Subspace Fitting Methods

In this section, the class of *subspace fitting* (SSF) methods is briefly presented. For a more detailed description, the reader is referred to [7]. The method which yields the most accurate estimates in this class is given special attention and is related to the statistically optimal maximum likelihood technique.

| Choice of M | Constraint on A | | |
|---|---|---|---|
| | $\mathbf{A} \in \mathcal{A}^d$ | $\mathbf{A} \in \mathcal{A}$ | $\mathbf{A} \in \mathcal{E}$ |
| $\mathbf{M}\mathbf{M}^* = \hat{\mathbf{R}}$ | Det-ML | Beamforming | ML-ESPRIT |
| $\mathbf{M} = \hat{\mathbf{E}}_s$ | MD-MUSIC | MUSIC | TLS-ESPRIT |
| $\mathbf{M} = \hat{\mathbf{E}}_s \mathbf{W}_{opt}^{1/2}$ | WSF | weighted MUSIC | weighted ESPRIT |

Table 1: Subspace fitting methods

### 3.1 Subspace Fitting Framework

The SSF methods are all based on fitting a representation of the data to a model in a least squares sense. Using the data matrix directly leads to the well-known *deterministic* (or conditional) maximum likelihood method [3, 4], herein referred to as the DML method. This is a non-linear least squares problem which, in general, requires a $d$-dimensional search procedure. When $d = 1$, the conventional beamforming method is obtained. The beamformer may also be used in the multiple emitter case by searching for $d$ isolated least squares fits in the one-dimensional criterion or – in more common terms – to search for $d$ peaks in the spatial spectrum.

As an alternative to the full data matrix, a low rank representation thereof can be used in the least squares fit. This leads to the class of subspace or eigendecomposition based methods. Using the left singular vectors of $\mathbf{X}_N$ that correspond to the $d'$ largest singular values (or equivalently, the $d'$ "largest" eigenvectors of $\hat{\mathbf{R}}$) results in a multidimensional version of the MUSIC (MUltiple SIgnal Classification) algorithm, [1, 2]. This technique was proposed in [5]. In the original MUSIC method, $d$ one-dimensional least squares fits are performed or, equivalently, $d$ peaks of the MUSIC spectrum are found.

Consider the general subspace fitting (SSF) problem

$$\min_{\mathbf{A},\mathbf{T}} \|\mathbf{M} - \mathbf{A}\mathbf{T}\|_F^2 \,. \tag{7}$$

The optimization (7) is a least-squares problem, which is linear in $\mathbf{T}$ and non-linear in $\theta$, since $\theta$ generally parametrizes $\mathbf{A}$ in a non-linear fashion.

By appropriate choices of data representation, $\mathbf{M}$, and array parametrization, $\mathbf{A}(\theta)$, several algorithms can be described by (7), see Table 1.

The following notation is used in the table: The eigendecomposition of the sample covariance is partioned according to

$$\hat{\mathbf{R}} = \hat{\mathbf{E}}_s \hat{\mathbf{\Lambda}}_s \hat{\mathbf{E}}_s^* + \hat{\mathbf{E}}_n \hat{\mathbf{\Lambda}}_n \hat{\mathbf{E}}_n^* \,, \tag{8}$$

where $\hat{\Lambda}_s$ is a diagonal matrix containing the $d'$ largest eigenvalues. The set of $d$-dimensional steering matrices is denoted

$$\mathcal{A}^d = \{\mathbf{A} \mid \mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_d)]\} \ . \tag{9}$$

Thus, the methods in the left column of Table 1 involve a multi-dimensional search – the cost function is minimized with respect to the signal parameters of all sources simultaneously. Introduce

$$\mathcal{A} = \{\mathbf{a} \mid \mathbf{a} = \mathbf{a}(\theta_i)\} \ . \tag{10}$$

This is identical to the previous parametrization but with the assumption that only one emitter is present. In this case (middle column of the table), the cost function is evaluated over a range of $\theta_i$ and the local extrema result in parameter estimates.

The right column contains a rather different parametrization which is specific to the ESPRIT algorithm, [6]. Although this technique will not be discussed further, it is included in the table to show the strength of the SSF framework. See [8] for more details. In the last row of Table 1, a weighting of the eigenvectors is introduced. The relevance of this weighting is discussed next.

### 3.2 The Weighted Subspace Fitting Method

For many estimation problems the maximum likelihood method provides the most accurate estimates possible, i.e., the Cramér-Rao Bound (CRB) is attained asymptotically in the number of data. This is not the case for the DML method, since the number of unknowns increase without bound when the sample size is increased. This opens up the possibility of finding an estimator that gives lower estimation error variance than does the DML technique. In [7], a general weighting of the signal eigenvectors is introduced, i.e., $\mathbf{M} = \hat{\mathbf{E}}_s\mathbf{W}^{1/2}$ is used in (7). A statistical analysis shows that the weighting which provides minimum variance estimates is the diagonal matrix

$$\mathbf{W}_{opt} = (\Lambda_s - \sigma^2\,\mathbf{I})^2\Lambda_s^{-1} \ . \tag{11}$$

The optimal SSF technique is termed the *weighted subspace fitting* (WSF) method. The optimal subspace weights are related to the inverse of the variance of each eigenvector in $\hat{\mathbf{E}}_s$. Thus, the weighting is particularly important when some of the eigenvector estimates are uncertain, as happens when the corresponding eigenvalue is close to the noise variance. This, in turn, is true when the signal covariance matrix is nearly singular due, for example, to multipath. In such scenarios, the WSF technique can give significantly more accurate estimates than the other SSF methods including the deterministic ML method.

For easy reference, the main steps of the WSF technique are summarized below.

**WSF Algorithm**

1. Form the sample covariance, $\hat{\mathbf{R}}$.

2. Calculate $\hat{\lambda}_k$, the eigenvalues of $\hat{\mathbf{R}}$. Find the $d'$ eigenvectors that correspond the largest eigenvalues and let these be the columns of $\hat{\mathbf{E}}_s$.

3. Estimate the noise variance by $\hat{\sigma}^2 = (\sum_{k=d'+1}^{m}\hat{\lambda}_k)/(m-d')$ . Form the weighting matrix $\hat{\mathbf{W}}_{opt} = (\hat{\Lambda}_s - \hat{\sigma}^2)^2\hat{\Lambda}_s^{-1}$.

4. Obtain the signal parameter estimates by solving

$$\hat{\theta} = \arg\max_{\theta} \mathrm{Tr}\left\{\mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\hat{\mathbf{E}}_s\hat{\mathbf{W}}_{opt}\hat{\mathbf{E}}_s^*\right\} .$$

The criterion function in step 4 above is obtained from (7) by minimizing explicitly with respect to the linear variable **T**. The indicated optimization problem can be solved by using a Newton-type descent technique, see [9] for details. The issue of initializing the search procedure and determining the numbers $d'$ and $d$ is also addressed in [9].

The previously mentioned DML method is based on a deterministic model of the emitter signals. If the latter are instead assumed to be temporally white and Gaussian distributed, the so-called *stochastic* ML (SML) method is obtained, [4]. For the Gaussian signal model the general theory of ML estimation is applicable, stating that the SML estimates are asymptotically efficient, i.e., the stochastic CRB is attained. However, also the SML procedure requires the solution of a multidimensional non-linear optimization problem. A Newton-type search for the SML technique can be implemented using $O(m^2d)$ flops per iteration, whereas the corresponding WSF implementation uses $O(md^2)$ flops, see [10]. In most problems of interest we have $d \ll m$, in which case the WSF optimization requires significantly less computing time. Furthermore, it has been shown that the WSF and SML estimates are asymptotically identical implying that they have the same estimation error covariance. See [11] for theoretical evidence and [10] for numerical experiments.

The asymptotic covariance matrix of the WSF and SML estimates can be expressed compactly as follows (for simplicity, only one signal parameter per source is assumed here)

$$\mathrm{E}\{(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T\} \simeq \frac{\sigma^2}{2N}\left(\mathrm{Re}\left\{(\mathbf{D}^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D})\odot(\mathbf{S}\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A}\mathbf{S})^T\right\}\right)^{-1} , \qquad (12)$$

where the matrix **D** contains the derivatives

$$\mathbf{D} = \left[\left.\frac{\partial \mathbf{a}}{\partial \theta}\right|_{\theta=\theta_1}, \ldots, \left.\frac{\partial \mathbf{a}}{\partial \theta}\right|_{\theta=\theta_d}\right] , \qquad (13)$$

$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$, and where $\odot$ denotes element-wise multiplication. This expression can be used to accurately predict the variance of the WSF estimates.

## 4 Experimental Results

Subspace based, high-resolution source localization techniques have been extensively analyzed on simulated array data. These studies show the potential of vast improvements in estimation accuracy and detection possibilities. Furthermore, multi-dimensional techniques such as WSF can cope with coherent multipath propagation which can cause severe problems for more traditional methods. Analysis of subspace based estimation procedures have shown that the resolution capabilities are only limited by the collection time and not by the physical antenna apperature, [7]. Of course, this is only true when modeling errors are not present. As the number of data increase, estimation errors due to model perturbations will dominate over errors due to noise, [12]. The application of subspace based techniques on real data is therefore of great interest.

Below, preliminary results from a project aimed at examining the viability of newly developed subspace based algorithms on hydro acoustic sensor array data are presented. The data is collected by a seven-element uniform linear array with an apperature of approximately one wavelength. The array is mounted horizontally and consists of sensors which are ideally omni-directional in the horizontal plane. A far field narrow band source is placed, in turn, in 18 different locations around array broadside. The purpose is to estimate the azimuthal angle to the emitter.

The data was digitized and stored for later off-line processing. In the figures below, the data is narrow band filtered and Hilbert transformed prior to forming the spatial covariance matrix, (6). Each estimate is based on three seconds of collection time and the SNR is approximately 6dB. Traditional beamforming is compared against WSF for 18 different emitter locations and the estimates are displayed in Figure 1. All values are normalized in terms of the 3 dB array beam width at broadside, (zero degrees). When estimating the principal eigenvector(s) of $\hat{\mathbf{R}}$, the noise is assumed to be spatially white, i.e., no pre-whitening is performed.

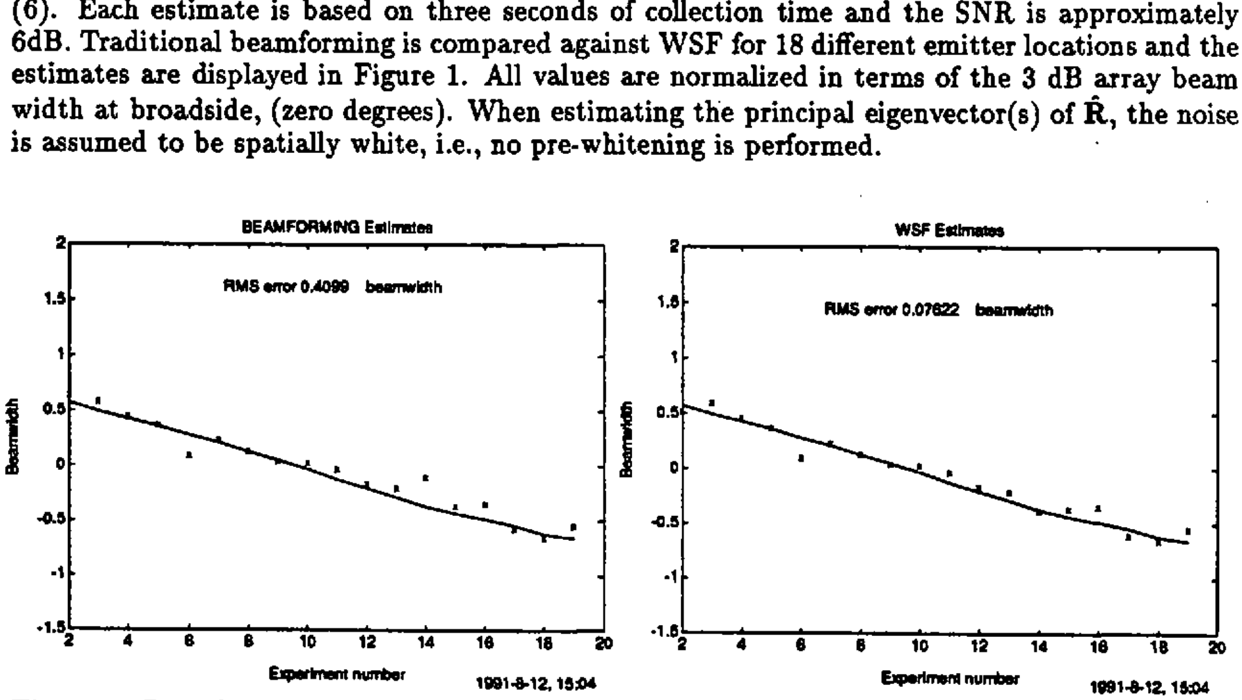


Figure 1: *Beamforming and WSF estimates for experiments 2–19. True emitter location indicated by solid line.*

Note that the root mean squared (RMS) error is substantially less for the WSF estimates as compared to the results obtained with beamforming. When one source is present, the beamforming estimates should not deviate significantly from the WSF estimates. However, multipath propagation is present in several of the experiments and two main propagation paths can be detected. In these cases, the WSF algorithm estimates two directions with high accuracy, whereas the beamforming estimates suffer. We expect that the performance differences will be even more pronounced when several emitters are active simultaneously in the same frequency band.

The errors seen above are predominantly modeling errors. The collection time is quite long and the effects of measurement noise is negligible. Two main sources of modeling errors may be expected. By examining the rank structure of the spatial covariance matrix, one notes that the noise is not spatially white. Secondly, the array elements are not identical resulting in a non-ideal array manifold. If the background noise structure is slowly varying, direct estimation of the noise covariance with no emitters is preferred. Estimating a parametrized noise covariance, [13], requires detailed knowledge of the noise characteristics. Modest array manifold errors, once characterized, can be handled by robust estimation techniques, [14, 15]. To obtain very accurate direction estimates, array calibration is often required. However, the treatment of calibration data is not a trivial problem.

It is of utmost importance to identify and characterize the type of modeling error encountered in underwater applications. With this knowledge, it may be possible to model systematic perturbations and formulate estimators which are less sensitive to unmodeled errors.

## 5 Conclusions

High-resolution signal parameter estimation techniques are herein formulated within a subspace fitting framework. Many so-called subspace based or eigenstructure techniques have a natural interpretation as a SSF problem. The optimal SSF method (termed WSF) and the traditional beamforming method are applied to real data, collected by a hydro acoustic array. The ideal response of a uniform linear array is assumed and fairly accurate azimuth estimates are obtained. The WSF estimates have approximately five times lower rms error than the corresponding beamforming results. The influence of modeling errors is discussed and possible techniques for handling model perturbations are mentioned.

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