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SUCCESSES AND FAILURES OF MULTIPLE-SCATTER THEORY

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INTRODUCTION

During the last five years there have been a number of important advances in multiple scatter theory. As a result encouraging agreement was obtained between the new theory and previously unexplained experimental data. In particular, the time spectra of acoustic intensity fluctuations caused by ocean internal waves [1], which failed to agree with the results of Rytov theory and its modifications, now fitted the predictions of the new theory well [2], [3], [4].

These initial successes have led to many papers aimed at extending and refining the theory. Much effort has also been devoted to investigating the accuracy of the theoretical expressions. In this context numerical simulations of random propagation have been particularly useful. Not only have they provided a check on theoretical results but have also given us much insight into properties of the wave-field fluctuations that cannot be obtained from ensemble average expressions.

In what follows we shall outline the new developments in theory and also describe the accompanying advances in numerical simulations. However, there are some important areas where theory still remains inadequate. For example, the cross-correlation of intensity fluctuations at different wave-frequencies cannot be satisfactorily described, nor can the probability distribution of the intensity fluctuations be derived theoretically.

ADVANCES IN THEORY

The principal quantities associated with wave-field fluctuations in a random medium that have prospects of being derived theoretically are the second and fourth moments, m_2 and m_4. The following equations for these moments have long been known [5], [6]

$$\frac{\partial m_2}{\partial z} = \frac{i}{2k}\left(\nabla_1^2 - \nabla_2^2\right)m_2 - \beta(1 - f_{12})\, m_2 \tag{1}$$

$$\frac{\partial m_4}{\partial z} = \frac{i}{2k}\left(\nabla_1^2 - \nabla_2^2 + \nabla_3^2 - \nabla_4^2\right)m_4 - \beta\left(2 + f_{13} + f_{24} - f_{12} - f_{23} - f_{14} - f_{34}\right)m_4 \tag{2}$$

where f_{ij} is the normalized two point autocorrelation function of refractive index fluctuations in the medium integrated in the direction of wave propagation z, k is the wave number of the radiation, ∇_i is the transverse Laplacian $\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}$, and β is the power attenuation coefficient of the random medium.

The second moment

The second moment m_2 describes the directional properties of the wave-field in the medium and also its mean intensity. Equation (1) for m_2 can be solved exactly for a number of useful cases including that of a point source. However, the introduction of a systematic refractive index profile in the medium leads to the appearance of curved ray paths and phenomena such as focussing, caustics and shadow zones. In this case the second moment equation cannot be solved exactly but various approximate methods such as eikonal series [7] and multi-scale expansions [8] have allowed these effects to be described with varying degrees of accuracy.

The fourth moment

The fourth moment m_4 describes the autocorrelation and variance of intensity fluctuations. The following solution of (2) was obtained in 1982 [9], for plane wave geometry, in the form of a multiple convolution for M, the Fourier transform of m_4

$$M(\nu,Z) = \frac{1}{(2\pi)^N} \iint \exp\{-\sum_{j=1}^{N} h(\xi_j,Q_j) + i\xi_j\,(\nu_j - \nu_{j-1})\}\, d\nu_1 \ldots d\nu_{N-1}\ ,\ d\xi_1 \ldots d\xi_N \quad (3)$$

where

$$h(\xi,Q) = 1 - f(\xi) - f(Q) + \tfrac{1}{2}f(\xi + Q) + \tfrac{1}{2}f(\xi - Q),$$

$$Q_j = [\nu_N + \nu_{N-1} + \ldots + \nu_j]\ Z/N,$$

$$N = 2\Gamma Z = 2\beta z.$$

This can be evaluated approximately to give the following useful expression

$$M_o(\nu,Z) = \frac{1}{2\pi} \int \exp\{-2\Gamma \int_o^Z h(\xi;\ \nu\,[Z - Z'])\, dZ'\}\ \exp\{-i\nu\xi\}\, d\xi\ . \quad (4)$$

A similar solution for a point source [2] and a medium that varied in time allowed a satisfactory explanation to be given for the acoustic intensity fluctuations observed in the Cobb Seamount experiment [3]. Rytov theory had failed to account for these.

This initial success provided the impetus for extending and refining fourth moment solutions. The next major step was to find a better estimate of the multiple convolution (3). This was done in 1983 [10] and the improved result gave very good agreement with numerical simulations.(see Fig. 1)

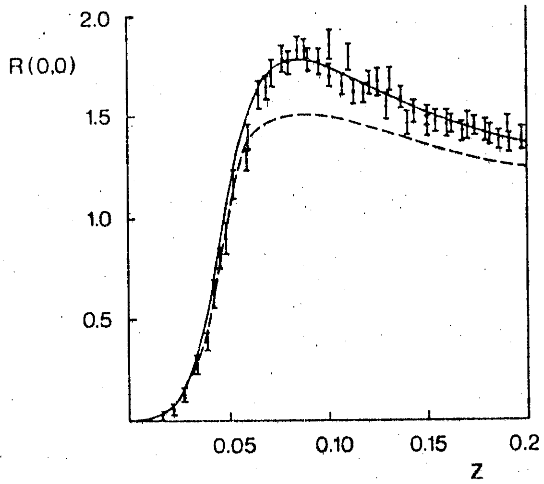

Fig. 1
Scintillation index for plane-wave geometry, $\Gamma = 1000$. The full curve gives the theoretical result and the points and error bars the results of numerical simulations.

Two-scale expansion

The original solution of the fourth moment equation was obtained by multiple convolution. However, other methods of solution were soon developed and it was encouraging to find that they yielded the same results as before, both as regards the fundamental approximation as well as the improved estimate. The first of the alternative methods used was that of the two scale expansion [10] which assumes that the required moment contains two distinct spatial scales. One of these is called the fast scale and the other the slow scale. Detailed investigations of the fourth moment solutions were carried out using the two scale method [11] and revealed much about the structure of the solutions.

Path integral methods

It had long been known that the various moments of the wave-field in a random medium can be written down in terms of the Feynman path integral [12]. This approach has the advantage that a medium with a deterministic refractive index profile can easily be dealt with, the ray paths arising naturally in the process of evaluation of the integral. This approach was used successfully to obtain the second moment m_2 [12]. It was subsequently used to find the fourth moment m_4 [13], giving results that were identical with those obtained by multiple convolution or two scale expansion. This extension of the fourth moment solution represents the most general and flexible form available at present. We shall now consider it in some detail.

General solution with profile

The general form of the fundamental approximation to the fourth moment in the case of a point source and an arbitrary deterministic refractive index profile is [13]

$$M_o(\nu,Z) = \frac{1}{(2\pi)^2} \iint \exp\left\{-2\Gamma \int_0^Z h(\underline{v}_1(Z') ; \underline{v}_2(Z'))\, dZ'\right\} \exp\left\{-ik \left[\underline{v}_2'(Z) \cdot \underline{v}_1(Z)\right]\right\} d\underline{v}_1(Z) \qquad (5)$$

where $\underline{v}_1$ and $\underline{v}_2$ are the solutions of the equations

$$\underline{v}_1''(Z') = \underline{v}_1(Z') \cdot \underline{W} \tag{5a}$$

$$\underline{v}_2''(Z') = \underline{v}_2(Z') \cdot \underline{W} + \underline{\nu} k\delta(Z - Z') \tag{5b}$$

$\underline{W}$ is a matrix of derivatives of n_o, the refractive index profile, along the ray path $\underline{S}$. Here $\underline{S}$ is the solution of the ray equation

$$\underline{S}''(Z') = \nabla n_o(\underline{S}(Z') \; ; \; Z') \tag{5c}$$

The following features of the general solutions are important:
(a) It describes a point source. In practice most sources are small and can be well approximated by a point.
(b) Because of the presence of the refractive index profile radiation from a point source follows the deterministic ray paths $\underline{S}(Z)$ in the absence of random irregularities. The $\underline{S}(Z)$ are found by solving the ray equation (5c). This is important in applications to ocean acoustics because of the sound speed profile that exists in all oceans. The solution (5) describes fluctuations due to scattering about a single ray path, but solutions can be combined to deal with convergence zones where several rays arrive along separate paths.
(c) The integration with respect to Z' in the exponent allows us to take into account variation in the scattering properties of the medium along the ray path. Such variations can occur in real situations and it is important to be able to deal with them.
(d) The ability to include curved ray paths allows us to deal with cases of foci and caustics.

Limitations of Cartesian coordinates

The obvious flexibility of the solution (5) means that it can be used in most cases encountered in practical ocean acoustics. A more accurate result can be achieved if necessary by the use of the next approximation which also handles points (a) - (d) above but has a more complicated structure. Certain serious objections have, however, been raised concerning the validity of the point source solutions described above. It has been suggested |14| that because the fourth moment was derived in a Cartesian system of coordinates the small departure from a truly circular wave front encountered in this representation leads to a large cumulative error in the multiple scatter limit, and that the solution could be grossly in error.

Curvilinear coordinate systems

A recent major advance in scattering theory, relevant to the objection mentioned above, has been the formulation of the moment equations in general curvilinear coordinate systems rather than in Cartesian coordinates |15|. In this way wave-fronts expanding from point sources can be dealt with exactly, while curved ray paths due to refractive index profiles can be made to coincide with the curvilinear coordinates and the limitations of small ray excursion and curvature can be removed. The only restriction now remaining is that the random irregularities in the medium should cause the radiation to depart by only small amounts from the deterministic ray paths.

The introduction of curvilinear coordinates has also enabled us to answer the question of the error arising when the wave-field from a point source is represented in Cartesian coordinates. The fourth moment equation, written in cylindrical coordinates, can be solved as a multiple convolution to give

$$M(\nu,Z) = \frac{1}{(2\pi)^N} \iint \exp\left\{-\sum_{j=1}^{N} h(R_j\,\Psi_j\,;\,R_j\,Q_j) + i\Psi_j\,(\nu_j - \nu_{j-1})\right\} d\nu_1 \ldots d\nu_{N-1}\;\; d\Psi_1 \ldots d\Psi_N \qquad (6)$$

where

$$Q_j = \frac{N}{R}\left[\nu_N\left(\frac{1}{N+1} - \frac{1}{N}\right) + \nu_{N-1}\left(\frac{1}{N} - \frac{1}{N-1}\right) + \ldots + \nu_j\left(\frac{1}{j+1} - \frac{1}{j}\right)\right]$$

and R is the scaled radial coordinate r.
Evaluation of (6) leads to results that are formally the same as those obtained when Cartesian coordinates are used, except that the z coordinate is replace by the radial coordinate r,

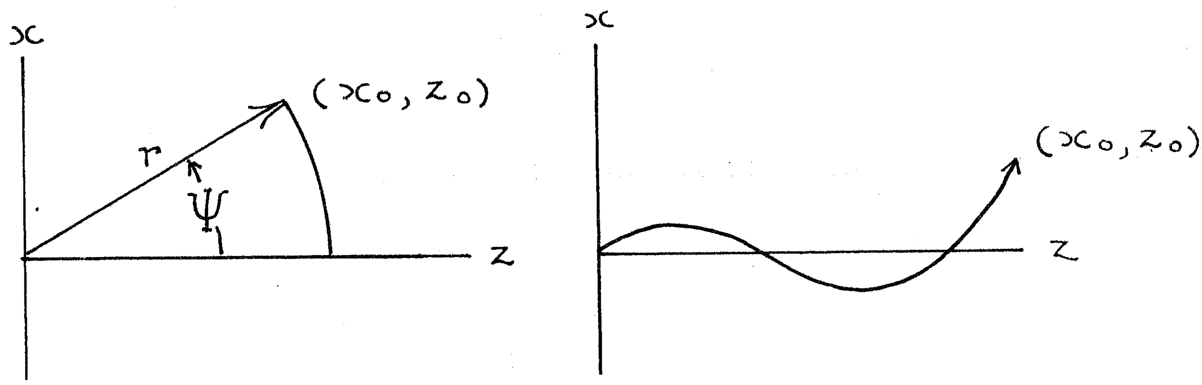

Fig. 2
(a) Cartesian and curvilinear coordinates for a point source with no refractive index profile.
(b) Case of a general refractive index profile.

This implies that the Cartesian representation is accurate to the extent that r can be replaced by z. In other words the point of observation (x_o, z_o) must remain close to the z axis. The same result presumably also holds true even when the presence of a refractive index profile leads to curved ray paths. It is important to keep this in mind when using existing fourth moment solutions. The three methods of solution discussed above have been carried out using a Cartesian coordinate system and so the results should only be used provided that the ray paths do not depart much from the z axis (the ratio x_o, z_o should be small). Solutions found in curvilinear coordinates would be free from such restrictions. However, apart from the case given above, these solutions have not yet been found. They will be a logical extension of the present work on fluctuation phenomena and will provide answers to some still unanswered questions in the field.

NUMERICAL SIMULATIONS

As a method for obtaining insight into fluctuation phenomena numerical simulations of random wave propagation area method that ranks equal in importance with the analytical approaches discussed above.

Plane wave geometry

Simulations have been used successfully to test the results of fourth moment theory in the case of plane wave geometry |16|. The principles of such numerical simulations are well known and will not be described here. A typical example of the intensity fluctuations induced in a plane wave travelling in a random medium containing irregularities with a Gaussian autocorrelation function is given in Fig. 3. The variance of these intensity fluctuations estimated from many such numerical simulations is shown by the points with error bars in Fig. 1. This provided a check on the accuracy of the theoretical expression shown in the same figure.

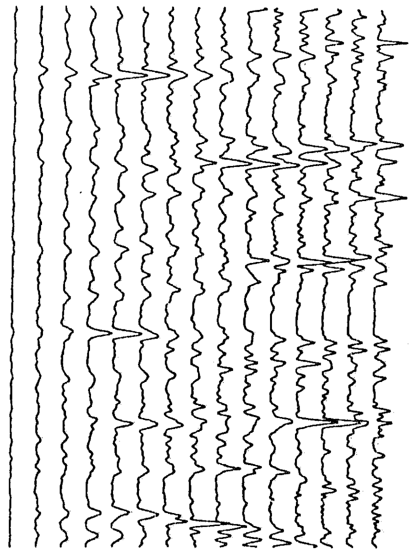

Fig. 3 Intensity fluctuations simulated for plane-wave geometry.

Point source

One method of checking the accuracy of the fourth moment solution in the case of a point source is by comparison with the results of numerical simulations, as was done in the case of plane wave geometry. However, numerical simulations involve considerable difficulties when we are dealing with a point source. This is because it is most convenient to represent the scattering medium in Cartesian coordinates. In such a coordinate system there are very rapid oscillations in the phase of the wave-field close to the source, and it is not possible to achieve the required resolution numerically. To overcome this a physically extended source |17| is used whose field approximates to that of a point source at ranges where the scattering effect is only just beginning to be important, and in a certain range of angles about the z axis. Simulations of intensity fluctuations using such a quasi-point source are shown in Fig. 4. The intensity fluctuation variance resulting from many such simulations is shown in Fig. 5

together with the theoretical variance obtained by solving the fourth moment equation for a point source. The agreement appears reasonable but there is a fairly large statistical scatter in the simulations. This is due to the fact that the intensity fluctuations are non-stationary transverse to the z direction and spatial averages cannot be used to substitute for ensemble averages. Thus the simulations must be repeated a much larger number of times in order to achieve a statistical average equivalent to that of the plane-wave case where spatial averages can also be used.

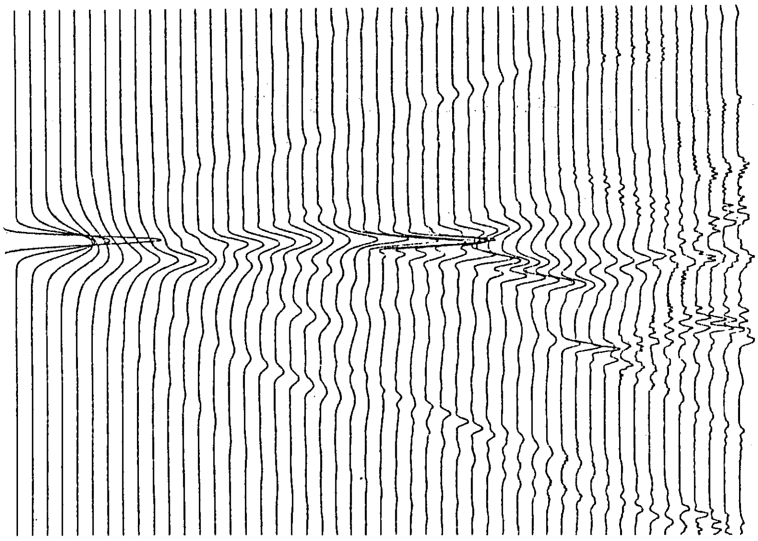

Fig. 4 Intensity fluctuations simulated with a quasi-point source.

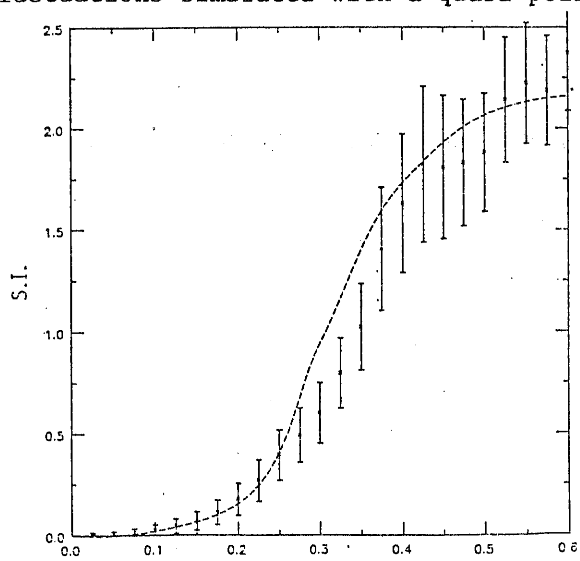

Fig. 5 Scintillation index from simulations with a quasi-point source. The full line gives the theoretical result.

Curvilinear coordinates can also be used in numerical simulations to avoid the problems associated with a point source [18]. Curved wave-fronts appear naturally in cylindrical coordinates and the rapid phase oscillations mentioned above are not encountered. Intensity fluctuations arising in the wave-field radiating from a point source are shown in Fig. 6.

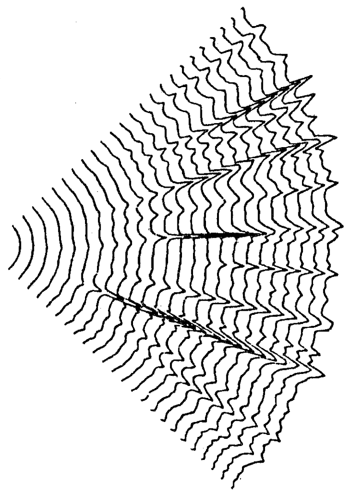

Fig. 6 Intensity fluctuations simulated with a true point source.

These fluctuations are now statistically stationary in the azimuthal direction and spatial averages can be used for ensemble averages. As a result better statistical averages can be obtained with fewer realizations. The variance of intensity fluctuations obtained by this method is shown in Fig. 7 together with the theoretical result, and the agreement is seen to be quite satisfactory.

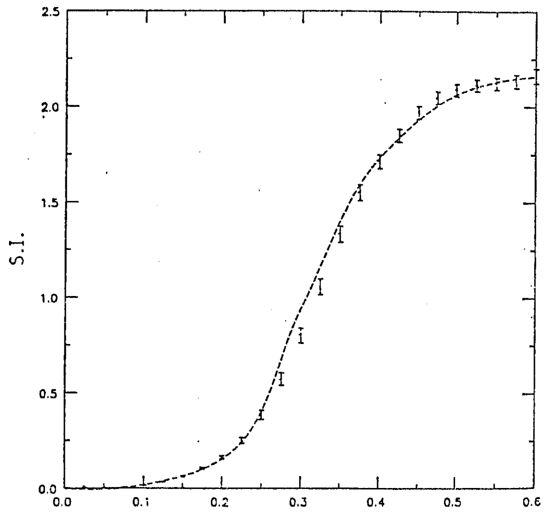

Fig. 7 Scintillation index from simulations with a true point source. The full line gives the theoretical result.

Thus numerical simulations have enabled us to verify the accuracy of our fourth moment solution in the case of a point source. We have additional confirmation of the conclusion, drawn above on theoretical grounds, that point sources can be adequately dealt with in Cartesian coordinate systems provided the observation point is not too far from the z axis.

Intensity cross-correlation

So far our review of scattering theory has been concerned with wave-fields of a single frequency. The record has been one of solid progress, theory having been extended into most areas of interest and the results substantiated by numerical simulations. The picture is not so encouraging when it comes to wave-fields at two different frequencies. The appropriate moment equations can be written down for the two-frequency case and a solution can be found for the fourth moment as a multiple convolution. However, attempts to evaluate this solution encounter serious difficulties |19|. It turns out that the resulting approximate intensity cross-spectrum does not agree with the exact cross-spectrum which can be obtained from the multiple convolution in the very far field |20|.

This failure is shown schematically in Fig. 8 where the approximate form is given by the full curves and the exact by the broken curves for two values of α the ratio of the wave-field frequencies. The further α departs from unity the worse the discrepancy becomes.

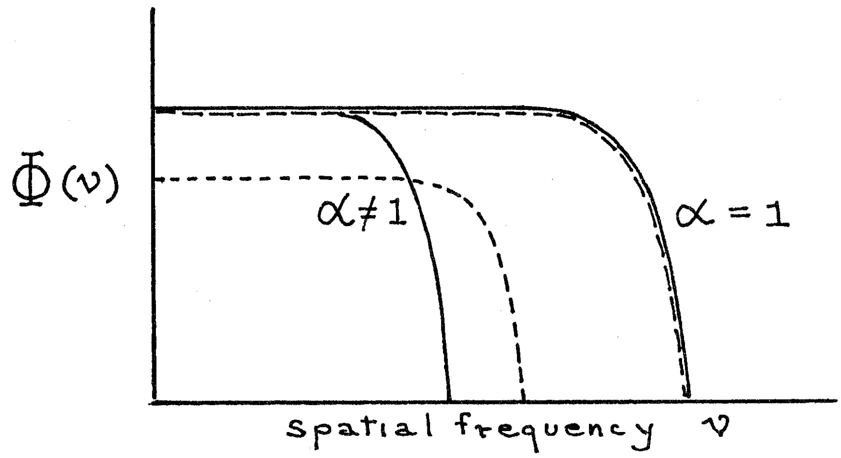

Fig. 8 Illustrating the failure of the cross-frequency spectrum of intensity fluctuations in the far field.

Detailed investigations |20| show that the intensity cross-spectra obtained from the fourth moment equation by existing standard methods can be used only for wave frequency ratios close to unity and for distances in the scattering medium that are not too large. All the methods discussed in the first part of the review suffer from the same draw-backs.

We conclude that some basically new approach to evaluating the fourth moment equation is required in the cross-frequency case. The cross-frequency correlation of intensity fluctuations stands out as one of the major unresolved problems in current random propagation theory.

Summary

Expressions for the first four moments of the wave field in a random medium can be now derived theoretically using a number of approaches. Point sources and refractive index profiles can be included. All these random wave propagation situations can be simulated using numerical methods.

Future developments should see the use of curvilinear coordinate systems combined with a variety of multiple scale expansions. This should allow such complicated phenomena as caustics and shadow zones to be dealt with in an increasingly exact manner.

Finally, the question of the cross-correlation of intensity fluctuations still remains to be resolved, and a theoretical derivation of the probability distributions of wave field fluctuations is still nowhere in sight.

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SCATTERING OF WAVES BY REFRACTIVE LAYERS WITH POWER LAW SPECTRA

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1. INTRODUCTION

Many useful insights into the phenomenology of wave propagation through extended inhomogeneous media have been obtained by studying the properties of a much simpler scattering system: the random phase changing screen. This system is of interest in its own right as a physical optics model for scattering by thin diffusing layers and rough surfaces both in transmissive and reflective geometries. In recent years, laboratory measurements of the scintillation of laser light scattered from turbulent plumes, mixing layers, mobile and rigid rough surfaces [1] have allowed more quantitative assessment of the predictions of a variety of theoretical phase screen models. As a result there has been renewed interest in the physical meaning of various statistical and spectral models and the mathematical implications of using them in phase screen calculations or indeed in the more complicated extended medium problem.

If, as is usually assumed, the phase distortions introduced by the screen constitute a Gaussian Process, then interest centres on the choice of phase autocorrelation function or spectrum. It is well known that autocorrelation functions which can be expanded in an even powered series about the origin, such as Gaussian or Lorentzian models, correspond to smoothly varying single scale phase functions which are infinitely differentiable. In the case of strong scattering, when the path fluctuations exceed a wavelength, non-Gaussian intensity patterns generated by screens of this type are dominated by geometrical optics effects [2]. On the other hand it is now recognised that raw power law models (ie without inner and outer scales) constitute the simplest class of multiscale screens. In this case the phase function is hierarchical, being self-affine under magnification and can be described in the language of Mandelbrot as a Gaussian random fractal [3]. Within this group of models further classification according to spectral index is necessary to distinguish between continuous functions which are not differentiable and those which are once, twice ... or n times differentiable. Clearly the physical implication of a model which is not differentiable will be the absence of geometrical optics effects: the predicted statistical properties of a scattered wave will include only the effects of diffraction and interference [4]. On the other hand a model which is only once differentiable will generate density fluctuations of geometrical rays but no caustics or focusing [5]. Evidently the spectral index and hence the truncation of differentiability determines the maximum order of singularity or catastrophe in the scattered wave field. In practice raw power law behaviour is not observed in nature: often regions of different power law index are found together with high and low frequency cut-offs (inner and outer scale effects). The presence of an outer scale always ensures that when sufficient area of the scatterer contributes to the wave field at the detector then Gaussian field statistics will be observed, the exact nature of the approach to this limit being determined by detail of the low frequency cut-off. The presence of an inner scale means that at sufficient magnification (small