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THE REDUCTION OF VIBRATION TRANSMISSION BY DESTRUCTIVE INTERFERENCE

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1. INTRODUCTION

Disturbances propagate through structures as waves, transmitting energy from one part to another. As they propagate the waves may be incident upon discontinuities which cause them to be partly reflected and partly transmitted. It is desirable to be able to predict the transmitted wave amplitudes so that the behaviour of a structure can be predicted. Also, in many cases it may be desirable to minimise the energy transmitted to one part of the structure from another in a certain frequency range.

This paper describes transmission through a split waveguide, by which is meant a waveguide, such as a beam or a taut string, which is divided into two parallel transmission paths. The intention is to introduce destructive interference between the component waves in each branch when the branches recombine. As such, it can be regarded as a more general structural analogue of Quincke's tubes [1]. The case of the split taut string is considered in some detail and the parameters which affect the transmission are identified. The case of beams in bending is also discussed. First, however, some relevant aspects of wave transmission are reviewed.

2. WAVE TRANSMISSION

We are here concerned with systems composed of essentially one-dimensional elements which act as waveguides. Wave components can propagate in both directions along the elements. There may be any number of distinct wave components depending on the nature of the system. For example, a taut string has one transverse wave component, a beam in bending has two (a propagating and a near field component), while a general structure may have flexural, torsional and axial wave components. Each component propagates with distance x as exp(-ikx), where k is the appropriate wavenumber and where harmonic time dependence exp(iωt) has been suppressed.

2.1 Scattering: Reflection and Transmission Matrices

In general, a structure will be composed of a number of uniform elements together with point discontinuities at the junction between two or more elements or where attachments are made. Waves propagating through the structure are reflected and transmitted at these discontinuities.

Consider (Fig. 1) a point discontinuity with positive $\boldsymbol{\Phi}^+$ and negative $\boldsymbol{\Phi}^-$ going waves at points 1 and 2 on either side of the discontinuity. As far as the discontinuity is concerned $\boldsymbol{\Phi}_1^+$ and $\boldsymbol{\Phi}_2^-$ represent incident or incoming waves while $\boldsymbol{\Phi}_2^+$ and $\boldsymbol{\Phi}_1^-$ are scattered or outgoing waves. They are related by

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$$\begin{Bmatrix} \Phi_1^- \\ \Phi_2^+ \end{Bmatrix} = \begin{bmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{bmatrix} \begin{Bmatrix} \Phi_1^+ \\ \Phi_2^- \end{Bmatrix} \qquad (1)$$

or $\quad \Phi_s = T\,\Phi_i \qquad (2)$

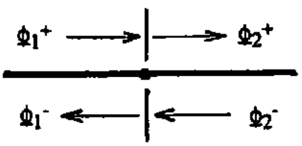

Fig. 1 Wave components at a discontinuity.

where T is a scattering matrix. It can be partitioned into matrices of transmission coefficients t_{12} and t_{21} and reflection coefficients r_{11} and r_{22}. Reflection and transmission matrices have broad application. Ref. [2] discusses some applications in structural dynamics, while [3] looks in more detail at beam vibrations and [4,5] discuss general approaches for more complex structures. If, on the other hand, points 1 and 2 in Fig. 1 are a distance l apart on a uniform waveguide, then

$$\Phi_2^+ = f\,\Phi_1^+ \; ; \; \Phi_1^- = f\,\Phi_2^- \qquad (3)$$

where f is a diagonal propagation matrix. Equivalently then $r_{11} = r_{22} = 0$ and $t_{12} = t_{21} = f$. The concept of a scattering matrix can be generalised to incoming and outgoing wave components between any two points on the waveguide and a method by which such scattering matrices can be determined is described in the next section. In general the scattering matrix T and the reflection matrices for a discontinuity or for some region of a structure will be square (same number of incident and scattered wave components) while the transmission matrices need not be. The eigenvalues of T will have magnitudes less or equal to one if the discontinuity is passive and the inverse of T need not exist.

In a practical context it may be desired to make some or all of the transmission coefficients zero, or at least small, over some frequency range so that one part of a structure can be isolated from disturbances which arise in another part of the structure.

2.2 Transmission Through Structures

When analysing structures transmission through more than one discontinuity or uniform section of waveguide can be found by combining the reflection and transmission matrices for the individual parts. For example, suppose in Fig. 2 the scattering matrices of elements a and b are known. The reflection and transmission matrices of a and b combined are

$$\begin{aligned} r'_{11} &= r^a_{11} + t^a_{12}\, D\, r^b_{22}\, t^a_{21} \\ t'_{13} &= t^a_{12}\, D\, t^b_{23} \\ t'_{31} &= t^b_{32}\, D\, t^a_{21} \\ r'_{33} &= r^b_{33} + t^b_{32}\, D\, r^a_{22}\, t^b_{23} \end{aligned} \qquad (4)$$

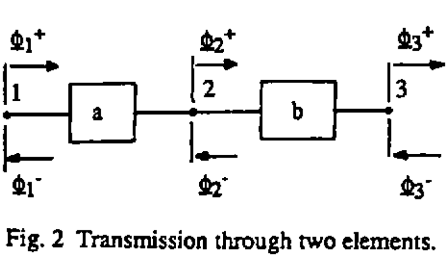

Fig. 2 Transmission through two elements.

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where the superscripts a, b, ' relate to elements a, b and a and b combined. Here

$$\mathbf{\Omega} = [\mathbf{I} - \mathbf{r}_{22}^{b}\, \mathbf{r}_{22}^{a}]^{-1} \qquad (5)$$

Physically, the internal wave components $\mathbf{\Phi}_2^+$ and $\mathbf{\Phi}_2^-$ are reflected at b, reflected at a, reflected at b again and so on, and $\mathbf{\Omega}$ represents the sum of all these successive reflections. When such interference is strongly constructive relatively large amplitude standing waves may exist at point 2, giving what may be termed an internal resonance.

2.3 The Split Waveguide

The split waveguide is shown in Fig. 3. A uniform waveguide divides into two propagation paths at point a and recombines at point b. While there are n wave components in $\mathbf{\Phi}_1^+$ and $\mathbf{\Phi}_4^+$ there will be 2n components in $\mathbf{\Phi}_2^+$ and $\mathbf{\Phi}_3^+$, n in each branch. The matrix $\mathbf{t}_{21}$, for example, will not be square.

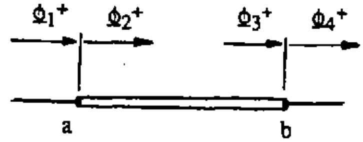

Fig. 3 The split waveguide.

The two branches of the split region will, in general, have different physical and geometric properties so that the wavenumbers in the branches are different. The transmitted waves $\mathbf{\Phi}_4^+$ for given incident waves $\mathbf{\Phi}_1^+$ can be found by the method discussed above. While this is generally a complicated procedure the means by which zero transmission can be achieved at discrete frequencies can be described, at least in qualitative terms, as follows.

Suppose that at a particular frequency the phase change experienced by the waves in both branches of the split region differ by π rad. When they recombine they will have opposite phase and destructive interference occurs, giving $\mathbf{\Phi}_4^+ = 0$. The incident waves $\mathbf{\Phi}_1^+$ will then be totally reflected. This is a good description for the case of taut strings, rather less so for the case of beams.

3. THE SPLIT STRING

The case where the waveguides are uniform strings under tension will now be considered. One transverse wave component can propagate in either direction along a string with a wavenumber $k = \omega\sqrt{\sigma/T}$ where σ is the mass per unit length and T is the tension in the string. The power flow in a wave of amplitude A is given by

$$\Pi = \frac{1}{2}(\omega A)^2 Z \qquad (6)$$

where $Z = \sqrt{\sigma T}$ is the characteristic impedance.

In Fig. 3, let the outer, upper and lower strings have wavenumbers k_1, k_2 and k_3 and characteristic impedances Z_1, Z_2 and Z_3. Let also the length of the split region be l. It is straightforward to show that the reflection and transmission matrices for the junction at a are

$$\underline{r}_{11} = (Z_1 - Z_2 - Z_3)/Z \quad ; \quad \underline{t}_{12} = \lceil Z_2 \;\; Z_3 \rfloor 2/Z$$
$$\underline{t}_{21} = \frac{2}{Z}\begin{bmatrix} Z_1 \\ Z_1 \end{bmatrix} \quad ; \quad \underline{r}_{22} = \frac{1}{Z}\begin{bmatrix} Z_2 - Z_3 - Z_1 & 2Z_3 \\ 2Z_2 & Z_3 - Z_2 - Z_1 \end{bmatrix} \tag{7}$$

where $Z = Z_1 + Z_2 + Z_3$. Similar results hold for junction b. The wave amplitudes $\underline{\Phi}_3^+$ are related to $\underline{\Phi}_2^+$ by the propagation matrix.

$$\underline{f} = \begin{bmatrix} \exp(-ik_2l) & 0 \\ 0 & \exp(-ik_3l) \end{bmatrix} \tag{8}$$

After some manipulation it can be shown that transmission across the split region is determined by the transmission coefficient.

$$t_{41} = \frac{-2i\,(\alpha_3 \operatorname{cosec} k_3l + \alpha_2 \operatorname{cosec} k_2l)}{(1 - i\,(\alpha_3 \cot k_3l + \alpha_2 \cot k_2l))^2 + (\alpha_3 \operatorname{cosec} k_3l + \alpha_2 \operatorname{cosec} k_2l)^2} \tag{9}$$

where $\alpha_2 = Z_2/Z$ and $\alpha_3 = Z_3/Z$. t_{41} depends on 3 parameters. The first is an interference parameter $\mu = k_2l/k_3l$ and equals the relative phase changes experienced by waves propagating in the two branches. When $\mu = 1$ no interference occurs. Interference is strongest generally when $\mu = 2$ as will be seen below. Branches 2 and 3 will be chosen such that $\mu \geq 1$. The second parameter $\alpha = \alpha_2/\alpha_3$ equals the ratio of impedances of the branches and indicates the "balance" of energies transported in each branch. The third parameter indicates the impedance mismatch $\beta = \alpha_2 + \alpha_3$ between the split region and the original string.

3.1 <u>The String With No Interference</u>
Suppose first that $\mu = 1$ so that no interference between the two branches takes place. The situation is identical to that where a uniform string changes in impedance by a factor β for a length l. This impedance mismatching gives rise to a transmitted wave of magnitude

$$t = |t_{41}| = \left[1 + \frac{\sin^2 k_3l}{4}\left(\beta - \frac{1}{\beta}\right)^2\right]^{-\frac{1}{2}} \tag{10}$$

as shown in Fig. 4. The frequency scale is given in terms of $\frac{k_3l}{2\pi} = l/\lambda$, where λ is the wavelength in branch 3. The transmission is a minimum when k_3l equals $\pi/2 + n\pi$, i.e. l is $\frac{1}{4}$ or $\frac{3}{4}$ of a wavelength. Successive internal reflections are here out of phase and give maximum destructive interference. When k_3l equals $n\pi$ however successive reflections are in phase, interfere constructively and this results in unity transmission. Note that substantial impedance mismatch is required to give substantial reductions in transmission.

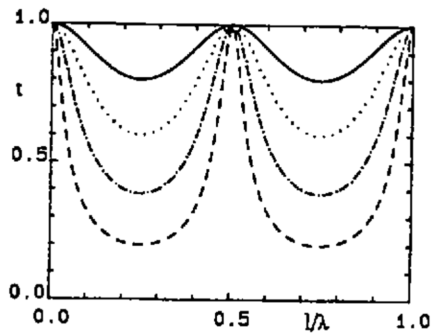

Fig. 4 Transmission through an impedance mismatch; $\mu = 1$:
—— $\beta = \frac{1}{2}, 2$; ····· $\beta = \frac{1}{3}, 3$;
-·-·-· $\beta = 0.2, 5$; ----- $\beta = 0.1, 10$.

3.2 The Effects of Interference

Figure 5 shows t for various μ with the branches equally balanced ($\alpha = 1$) and no impedance mismatching ($\beta = 1$). Zeros occur for values of $k_3l/2\pi$ such that

$$\frac{\sin \mu k_3 l}{\sin k_3 l} = -\alpha \tag{11}$$

For $\alpha = 1$ the roots of this equation are given by

$$\frac{l}{\lambda} = \frac{n}{(\mu + 1)}, \frac{n - \frac{1}{2}}{(\mu - 1)} ; n = 1, 2, 3... \tag{12}$$

In the range of l/λ up to 1 there are two zeros for $\mu \leq 1.5$, three for $\mu \leq 2$, four for $\mu \leq 2.5$ and so on.

The zeros of t indicate the frequency range in which the interference of waves in the two branches causes the transmission to be reduced. Between the zeros t rises to a peak. Note that for $\mu = 2$, the three zeros are equally spaced and that t is symmetrical, the peaks between the zeros having the same height. This value for μ then gives the widest frequency range in which t is below a certain value and hence in this respect indicates when these interference effects are strongest.

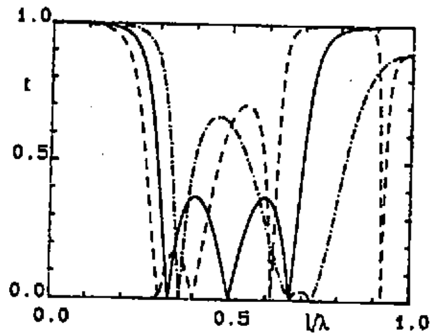

Fig. 5 Effects of interference; $\alpha = \beta = 1$:
-·-·-· $\mu = 1.75$; —— $\mu = 2$;
----- $\mu = 2.25$.

It is instructive to note that at each frequency of zero transmission the wave amplitudes in the two branches are in the ratio $(-\alpha_3/\alpha_2)$, while the internal force amplitudes are equal and opposite and the powers carried in each direction in each branch are in the ratio (α_3/α_2). The net power flow is zero since the branches carry standing waves. The displacement of the system is as indicated in Fig. 6. Point b is fixed, equilibrium being maintained by the equal and opposite forces in the two branches. The incident wave is reflected at point a with a reflection coefficient whose magnitude is unity, so a standing wave also exists on this part of the system.

3.3 The Effects of Balance

Figure 7 shows the effect of varying α with $\mu = 2$ and $\beta = 1$. Zeros now occur when

$$k_3 l = \pi, \pi \pm \cos^{-1}(\alpha/2) \qquad (13)$$

Thus, as α increases the three zero frequencies move together, the height of the peaks between them decreasing until they coalesce at $\alpha = 2$. The frequency range where transmission is reduced thus gets smaller as α increases, but the transmission within this band is also reduced.

3.4 The Effects of Impedance Mismatch

Figure 8 shows the case where $\mu = 2$, $\alpha = 1$ with various β and therefore illustrates the effects of impedance mismatch between the sum of the split branches and the main string. Note that the zero transmission frequencies are fixed since α is constant. As was seen in section 3.1 the impedance mismatch typically affects transmission at frequencies where l is $\frac{1}{4}$ or $\frac{3}{4}$ of a wavelength. Similar effects were observed in Fig. 4 where no interference took place, but where much greater impedance mismatches were considered. Transmission in the interference region, where l/λ is around $\frac{1}{2}$, is not affected greatly.

3.5 Transmission with Branch 3 Constant

Finally, Fig. 9 illustrates the case where $\alpha_3 = 1$ and $\mu = 2$ with various α_2. This corresponds to the situation where a side branch is attached to a uniform string. Since α_2/α_3 and $\alpha_2 + \alpha_3$ both vary the zero frequencies are not fixed and impedance mismatching effects are present. Certain fixed frequencies are apparent, where t is

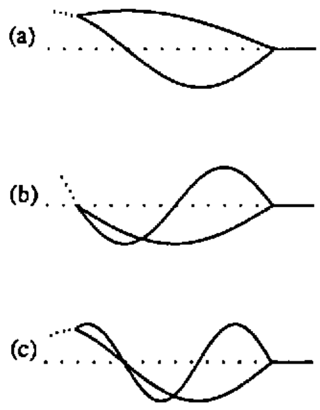

Fig. 6 Standing wave amplitudes; $\mu = 2, \alpha = 1, \beta = 1$:
(a) first zero of t,
(b) second zero, $k_3 l = \pi$,
(c) third zero.

Fig. 7 The effects of balance between the two branches; $\mu = 2, \beta = 1$:
----- $\alpha = 0.5$; —— $\alpha = 1$;
····· $\alpha = 1.5$; -·-·-· $\alpha = 2$.

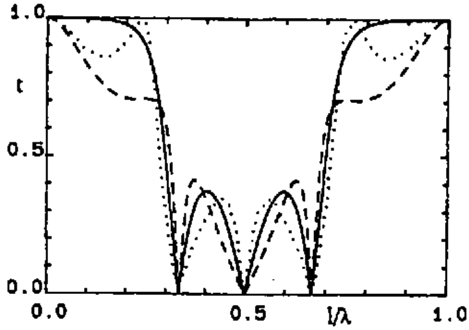

Fig. 8 The effects of impedance mismatch in the presence of interference: $\mu = 2, \alpha = 1$: ····· $\beta = 0.5$; —— $\beta = 1$; ----- $\beta = 2$.

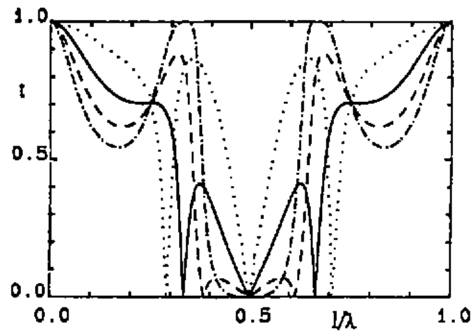

Fig. 9 The effects of balance with branch 3 constant: $\mu = 2, \alpha_3 = 1$; ····· $\alpha = 0.5$; —— $\alpha = 1$; ----- $\alpha = 1.5$; -·-·-·- $\alpha = 2$.

independent of α_2. These occur at those frequencies where $k_3 l = n\pi/\mu$ and here

$$t^2 = \sin^2 k_3 l / \left[\sin^2 k_3 l + \alpha_3{}^2 (1 - (-1)^n \cos k_3 l)^2\right] \tag{14}$$

The frequencies of zero transmission occur between these fixed frequencies. For the case where $\alpha_2 = \alpha_3$ these fixed frequencies are also points of inflection.

4. CONCLUDING REMARKS

In the previous section it was demonstrated how interference between the waves in the two branches of the split waveguide can result in low transmission over a wide range of frequencies. An impedance mismatch alone (Fig. 4) can equally give low transmission but only for substantial values of the mismatch. These have resulting consequences for the stiffness or mass of this part of the structure which may be undesirable. The split waveguide, however, operates with little change to the mass and stiffness. The physical length is around a wavelength of the branch with the largest wavenumber. The bandwidth depends on the impedance ratio of the two branches. If the branches are of very dissimilar impedance then transmission is reduced only in fairly narrow frequency ranges. Transmission through more complex arrangements can be determined but generally interpreting the results can be difficult.

The case of flexural vibrations is different in the respect that there are two wave components, one propagating and one near field, two distinct internal forces, namely bending moment and shear, and two variables, displacement and rotation, describing the motion of the beam. The nearfields increase numerical complexity. In the split region the transmission and reflection matrices are 4 x 4 while the overall transmission is described by a 2 x 2 matrix $\underline{t}_{41}$. The effect of this is to make analytical study difficult, but the basic phenomena are the same as the string: transmission is governed by the wavenumber ratio and the relative impedances.

Now, however, there are two mechanisms for transporting energy along the beam: moment/rotation and shear force/displacement. Full isolation is achieved when $\underline{t}_{41} = 0$. Then the right hand side of the split region is effectively clamped and no power can flow through it. However, it can be shown that this condition cannot be achieved with two branches (although it can be achieved with three branches). In practice this need not be a problem since it would usually suffice to make the (1,1) element of $\underline{t}_{41}$ zero thus ensuring that the propagating component of the transmitted waves is zero. Only a near-field would exist to the right of the split section where the moment, slope, shear force and displacement would all be in phase, thus transmitting no power.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

[1] LORD RAYLEIGH, 'The Theory of Sound', Vol. 2, London: McMillan (1877).
[2] L CREMER, M HECKL & E E UNGAR, 'Structure-Borne Sound', (2nd ed.), Berlin: Springer-Verlag (1988).
[3] B R MACE, 'Wave Reflection and Transmission in Beams', *Journal of Sound and Vibration*, 97, 237-246 (1984).
[4] D W MILLER & A H von FLOTOW, 'A Travelling Wave Approach to Power Flow in Structural Networks', *Journal of Sound and Vibration*, 128, 145-162 (1989).
[5] Y YONG & Y K LIN, 'Propagation of Decaying Waves in Periodic and Piecewise Periodic Structures of Finite Length', *Journal of Sound and Vibration*, 129, 99-118 (1989).