WAVE TRANSMISSION AND REFLECTION IN BEAMS WITH DISCONTINUITIES

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1. INTRODUCTION

A time harmonic flexural wave which propagates along a uniform beam in the positive x-direction can be written as $w_0 \exp i(\omega t-kx)$ where the wavenumber $k = (m\omega^2/EI)^{\frac{1}{4}}$. It is well known that when such a wave strikes a discontinuity it gives rise to propagating reflected $(\exp i(\omega t+kx))$ and transmitted $(\exp i(\omega t-kx))$ waves together with non-propagating reflected $(\exp(i\omega t+kx))$ and transmitted $(\exp(i\omega t-kx))$ waves, the so-called near field. The amplitudes of the various components can be expressed in terms of the amplitude of the incident wave and transmission and reflection coefficients which depend on the properties of the discontinuity itself [1]. The amplitude of the near-field wave attenuates rapidly with distance such that it decreases by a factor of about 500 in one wave length. The effects of the near field can thus be neglected at sufficiently large distances. However, if an applied force or a second discontinuity is close enough to the first such that the amplitude of the near field is not negligible then this near field component can give rise to significant transmitted and reflected waves of both the propagating and non-propagating kinds. In this paper the reflection and transmission coefficients for both propagating and near field waves will be given.

In any situation we must satisfy the differential equation of motion for the beam, the boundary conditions at the discontinuities and any other boundary conditions that may be imposed. The use of reflection and transmission coefficients will often simplify the analysis.

2. TRANSMISSION AND REFLECTION COEFFICIENTS

Consider a uniform beam with a point discontinuity at $x=0$. This discontinuity may be a mass, spring or damper or some combination of these and may in general exert both a rotational and a translational constraint on the beam. For simplicity supports which exert only a translational constraint will be considered here. The support will be represented by a translational spring of stiffness K_T which may be positive, negative or complex depending on the nature of the support. If a wave of unit amplitude propagates along the beam in the positive x-direction then the displacement of the beam can be written as

$$\begin{aligned} w(x) &= e^{-ikx} + Re^{ikx} + R_N e^{kx} \qquad x \leqslant 0 \\ w(x) &= Te^{-ikx} + T_N e^{-kx} \qquad x \geqslant 0 \end{aligned} \tag{1}$$

where the time dependence $\exp(i\omega t)$ has been suppressed. R and T are the reflection and transmission coefficients for the propagating waves and R_N and T_N are the coefficients for the near field. Since the support exerts no rotational constraint on the beam then we can assume that the displacement, slope and bending moment are continuous at $x=0$. This leads to three equations in the coefficients. A fourth equation can be found by considering the shear force in the beam at $x=0$. Solving these equations gives

$$T = \cos\psi\, e^{i\psi} \qquad T_N = \sin\psi\, e^{i\psi}$$
$$R = i\sin\psi\, e^{i\psi} \qquad R_N = \sin\psi\, e^{i\psi} \tag{2}$$

where the parameter ψ is given in terms of the stiffness of the support K_T by

$$\cot\psi = \frac{4EIk^3}{K_T} - 1 \tag{3}$$

For real K_T, ψ lies in the range $(-\pi/2, \pi/2)$, is zero if $K_T=0$ and tends to $-\pi/4$ as $K_T \to \infty$. The transmitted wave (T) thus has a positive component in phase with the incident wave and lags the reflected wave by $\pi/2$. The near field waves are of equal amplitude and phase. The energy associated with each propagating wave is proportional to the square of the amplitude and thus if K_T and hence ψ are real we see that $|T|^2 + |R|^2 = 1$. Thus energy is conserved: the energy in the incident wave equals the sum of the energies in the transmitted and reflected waves.

Now let us consider the case where the incident wave is a near field wave e^{-kx}. The displacement of the beam $w(x)$ can now be written as :

$$w(x) = e^{-kx} + R^A e^{ikx} + R_N^A e^{kx} \qquad x \leq 0$$
$$w(x) = T^A e^{-ikx} + T_N^A e^{-kx} \qquad x \geq 0 \tag{4}$$

where the superscript A denotes that the incident wave is of the attenuating type. By the same considerations we find that

$$T^A = i\sin\psi\, e^{i\psi} \qquad T_N^A = 1 + \sin\psi\, e^{i\psi}$$
$$R^A = i\sin\psi\, e^{i\psi} \qquad R_N^A = \sin\psi\, e^{i\psi} \tag{5}$$

Now the reflected and transmitted waves are in phase and of equal magnitude. The energy that propagates away from the support is

$$E = |T|^2 + |R|^2 = 2\sin^2\psi \tag{6}$$

and thus depends on the nature of the support. Equations 5 and 6 imply therefore that a near field wave incident upon a support can give rise to substantial propagating transmitted and reflected waves.

These results can be neatly summarised in matrix form. Let $\underset{\sim}{A}^+$ be the 2x1 vector of positive going incident waves at $x=0$ (i.e. the amplitudes of the e^{-ikx} and e^{-kx} waves), $\underset{\sim}{B}^+$ be a similar vector for the transmitted waves and $\underset{\sim}{A}^-$ represent the negative going reflected waves (Fig. 1). $\underset{\sim}{B}^+$ and $\underset{\sim}{A}^-$ are related to $\underset{\sim}{A}^+$ by reflection and transmission matrices $\underline{R}$ and $\underline{T}$ such that

$$\underset{\sim}{B}^+ = \underline{T}\,\underset{\sim}{A}^+ \qquad \underset{\sim}{A}^- = \underline{R}\,\underset{\sim}{A}^+ \tag{7}$$

where

$$\underline{T} = \begin{bmatrix} T & T^A \\ T_N & T_N^A \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} R & R^A \\ R_N & R_N^A \end{bmatrix} \tag{8}$$

and from equations 2 and 5 we see that

$$\underline{T} = \underline{I} + \mu\,\underline{C} \qquad R = \mu\,\underline{C} \qquad \mu = \sin\psi\, e^{i\psi} \qquad \underline{C} = \begin{bmatrix} i & i \\ 1 & 1 \end{bmatrix} \tag{9}$$

If the discontinuity also provides a rotational constraint a similar analysis can be performed, the expressions for $\underline{T}$ and $\underline{R}$ then containing an extra term.

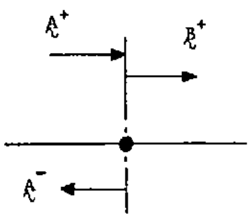

FIG. 1. Wave vectors on beam with single discontinuity.

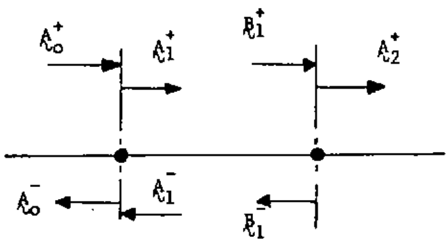

FIG. 2. Beam with two discontinuities.

3. TRANSMISSION THROUGH TWO DISCONTINUITIES

In the previous section the transmission and reflection matrices $\underline{T}$ and $\underline{R}$ were given, and it was seen that near field waves incident upon a discontinuity can give rise to propagating waves. As an example of a case where this near field interaction is important consider a beam with two identical discontinuities a distance x apart (Fig. 2). A propagating wave, originating from some source situated to the left, is incident upon the first discontinuity, and we may wish to determine the ratio of transmitted energy to incident energy $\tau = |A_2^+|^2/|A_o^+|^2$, where A_2^+ and A_o^+ are the propagating components of the wave vectors $\underset{\sim}{A}_2^+$ and $\underset{\sim}{A}_o^+$. The transmission efficiency τ can be found in a number of ways but the use of the transmission and reflection matrices helps to reduce the amount of algebraic manipulation often associated with this kind of problem. Now from the definitions of $\underline{T}$, $\underline{R}$ and the wave vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ we have

$$\underset{\sim}{A}_1^+ = \underline{T}\,\underset{\sim}{A}_o^+ + \underline{R}\,\underset{\sim}{A}_1^- ; \qquad \underset{\sim}{B}_1^- = \underline{R}\,\underset{\sim}{B}_1^+ ; \qquad \underset{\sim}{A}_2^+ = \underline{T}\,\underset{\sim}{B}_1^+ ;$$

$$\underset{\sim}{B}_1^+ = \underline{F}\,\underset{\sim}{A}_1^+ ; \qquad \underset{\sim}{A}_1^- = \underline{F}\,\underset{\sim}{B}_1^- ; \qquad \underline{F} = \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{-kx} \end{bmatrix} \tag{10}$$

The solution of these equations and the determination of τ is quite straightforward. τ lies between 0 and 1 and is equal to 1 at the

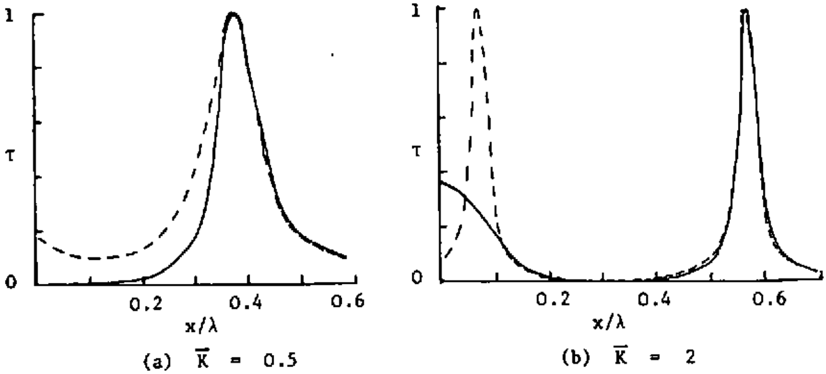

FIG. 3. Transmission efficiency against support spacing, (----) neglecting and (——) including near fields.

resonances of the internal section of beam and the two supports. At these resonances all the incident energy is transmitted through the supports. It is obvious that τ depends on the spacing x of the supports. Figure 3 shows this dependence for the cases where the dimensionless support stiffness $\overline{K} = K_T/4EIk^3 = 0.5$ and 2. The curves show τ as a function of support spacing compared to that value calculated by neglecting the near fields between the supports. It is clear that for spacings greater than say one half of a wavelength the two results are within a few percent. This is expected since the near field decays so rapidly that we are justified in neglecting it at such distances. However, for closer spacings than this the effects of the near fields between the supports on the transmission of energy are important and it is apparent that these near fields should be included at such spacings.

4. CONCLUDING REMARKS

The solution to problems of wave transmission on beams which lie on supports is facilitated by the use of transmission and reflection coefficients. The traditional method of considering the propagating components alone is applicable when the supports are far enough apart so that the near fields of one support can be neglected at neighbouring supports. A simple set of simultaneous equations has then to be solved. When the near fields cannot be ignored their effects can be dealt with by considering transmission and reflection matrices. The solution can then be obtained by considering a straightforward set of simultaneous 2x2 matrix equations.

REFERENCE

1. E.E. UNGAR 1961 Journal of the Acoustical Society of America 33, 633-639. Transmission of Plate Flexural Waves through Reinforcing Beams; Dynamics Stress Concentration.