# Inverse Scattering and Field Extrapolation for Rough Surface Scattering

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## 1 Introduction

In this paper we propose a new algorithm for field extrapolation and inverse scattering by rough surfaces. The method is based on the method by Potthast [7] for inverse scattering by a bounded obstacle and requires measurements of the acoustic field on a finite line above the scattering surface. Formulae, based on reciprocity arguments, and requiring solution of a first kind integral equation by Tikhonov regularization, are proposed for constructing the acoustic field up to the boundary, in the time and frequency domains. Theoretical results justifying these formulae are summarised and the effectiveness of the algorithms demonstrated by numerical simulations. We explain how extrapolation of the field up to the boundary can be used as the basis of an inverse scattering algorithm, identifying the surface position from measurements of the scattered field, and illustrate this with numerical experiments.

## 2 The Surface Scattering Problem

We adopt Cartesian coordinates $Ox_1x_2x_3$ and assume throughout that the scattering surface, denoted by $\Gamma$, is invariant in the $x_3$-direction, having the equation $x_2 = f(x_1)$ for some given bounded and continuously differentiable function $f$. To illustrate the principle of the method we propose, we consider in this paper the 2D case in which the acoustic source is an incoherent line source, parallel to the $x_3$-axis, so that both incident field and scattering surface are invariant in the $x_3$-direction. We will present results later for the case of an impulsive source, but for the moment consider the behaviour of a single mono-frequency component, so that the source is time harmonic ($e^{-i\omega t}$ time dependence). The method we propose in Section 3 for extrapolating the measured field up to the boundary is independent of the boundary condition. To make things specific, however, we describe the method for the case when the scattering surface is sound soft, so that the pressure vanishes on $\Gamma$. The acoustic pressure satisfies the wave equation in the region $D$ where $x_2 > f(x_1)$ above $\Gamma$. With these assumptions, and letting $x$ and $y$ denote the vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the acoustic pressure at $(x_1, x_2, x_3)$ when the source occupies the line $x_1 = y_1$, $x_2 = y_2$, will be denoted by $G(x, y)$. Then the function $u$, defined by $u(x) := G(x, y)$, satisfies the inhomogenous Helmholtz equation,

$$\Delta u + k^2 u = -\delta_y \tag{1}$$

in $D$, with $k = \omega/c$ and $c$ the speed of sound. Further, $u = 0$ on $\Gamma$ and $u$ satisfies the Sommerfeld radiation conditions (equations (2) and (3) below).

Let $\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x-y|)$. Then $\Phi(\cdot, y)$ is the field at $x$ when the line source is at $y$ in free-field conditions, and so is the field incident on the scattering surface. Let $U^s(x, y) := G(x, y) - \Phi(x, y)$ denote the scattered part of the acoustic field. Then $v(x) = U^s(x, y)$ satisfies the following Dirichlet boundary value problem (the direct problem): find $v \in C^2(D) \cap C(\overline{D})$ such that

$$\Delta v + k^2 v = 0 \quad \text{in } D,$$

$$v = -\Phi(., y) \quad \text{on } \Gamma,$$

$$v(x) = O(r^{-\frac{1}{2}}), \tag{2}$$

$$\frac{\partial v(x)}{\partial r} - ikv(x) = o(r^{-\frac{1}{2}}), \tag{3}$$

as $r := |x| \to \infty$, uniformly in $\hat{x} = x/|x|$. From [3] and [2, Theorem 5.1], we have that this boundary value problem has exactly one solution.

For some $h \in \mathbf{R}$ let $U_h$ be the region where $x_2 > h$ and $x_1 \in \mathbb{R}$, and define

$$G_{1,h}(x,y) := \Phi(x,y) + \Phi(x,y'_h) + P(k(x-y'_h)), \qquad x,y \in \overline{U_h}, \;\; x \neq y,$$

where

$$P(z) := \frac{e^{i|z|}}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}} e^{-|z|t}(1+\gamma(1+it))}{\sqrt{t-zi}(t-i(1+\gamma))^2} \,\mathrm{d}t, \quad z \in \overline{U_0},$$

with $\gamma := z_2/|z|$, and $y'_h := (y_1, 2h - y_2)$ the reflection of $y$ in the straight line $\Gamma_h := \partial U_h$. Thus $G_{1,h}$ is the Green's function for the Helmholtz equation in the half-plane $U_h$ which satisfies the impedance boundary condition, $\frac{\partial G_{1,h}(x,y)}{\partial x_2} + ikG_{1,h}(x,y) = 0$. From [1], $G_{1,h}$ also satisfies the bound

$$|G_{1,h}(x,y)| \leq C(1+x_2-h)(1+y_2-h)|x-y|^{-\frac{3}{2}}, \quad x,y \in \overline{U_h}, \;\; x \neq y, \tag{4}$$

for some constant $C > 0$ depending only on $k$, and so $G_{1,h}$ decays faster than required by the Sommerfeld radiation conditions if $x_2$ and $y_2$ stay close to the boundary $\Gamma_h$.

Therefore, echoing earlier notation we define $U_1^s(x,y) := G(x,y) - G_{1,h}(x,y)$ for $x,y \in \overline{D}$, so that $U_1^s(\cdot,y)$ is the solution to the above Dirichlet BVP for boundary data $-G_{1,h}(\cdot,y)$ on $\Gamma$.

## 3 The Inverse Problem

Let $A > 0$, $H > \sup f$, and let $\gamma^*$ denote the finite horizontal line of height $x_2 = H$ for $|x_1| \leq A$. The inverse problem we consider is the following:

*Given measurements of the total field $G(x,z)$, for $x \in \gamma^*$ and a single $z \in D$, determine $f$, i.e. the location of the infinite surface $\Gamma$.*

Suppose that $x^* \in D\backslash\{z\}$ and $f_0$ is a bounded and continuously differentiable function such that $f_0(0) < 0$. Define the translated function $f_{x^*}$ by $f_{x^*}(x_1) := f_0(x_1 - x_1^*) + x_2^*$, and denote $\Gamma_{x^*}$ by having the function $x_2 = f_{x^*}(x_1)$. Since $f$ is a bounded function it holds that, for some real constants $f_-, f_+, f_- \leq f(x_1) \leq f_+$, for $x_1 \in \mathbf{R}$. Assuming that $f_-$ and $f_+$ are known, choose $d > \max(f_+ - f_-, \epsilon)$ where $\epsilon := \inf_{x_1 \in \mathbf{R}} f_0(x_1)$. We will solve approximately, by Tikhonov regularization with regularization parameter $\alpha$, the equation

$$\int_{\gamma^*} G_{1,h^*}(x,y)\phi_{x^*}(y)\,\mathrm{d}s(y) = g_{x^*}(x) := G_{1,h^*}(x,x^*), \qquad x \in \Gamma_{x^*}, \tag{5}$$

where $h^* := x_2^* - d$, obtaining the approximate solution $\phi_{x^*}^\alpha$. This equation can be written in operator form as $K\phi_{x^*} = g_{x^*}$, for $\phi_{x^*} \in L^2(\gamma^*)$. Note that, in view of the bound (4), $g_{x^*} \in L^2(\Gamma_{x^*})$, and further $K : L^2(\gamma^*) \to L^2(\Gamma_{x^*})$ and is bounded, (see [4]). It is shown in [5] that $K$ is injective, has dense range and following this that $\|K\phi_{x^*}^\alpha - g_{x^*}\|_{L^2(\Gamma_{x^*})} \to 0$ as $\alpha \to 0$, though also $\|\phi_{x^*}^\alpha\|_{L^2(\gamma^*)} \to \infty$ as $\alpha \to 0$ since $g_{x^*}$ is not in the range of $K$.

Consider the duct-like region $G^*$ between $\Gamma_{x^*}$ and $\Gamma_{h^*}$ i.e. $h^* < x_2 < f_{x^*}$ and let the residual in (5) be defined by

$$u(x) := \int_{\gamma^*} G_{1,h^*}(x,y)\phi_{x^*}^\alpha(y)\mathrm{d}s(y) - G_{1,h^*}(x,x^*), \qquad x \in \overline{G^*}.$$

Note we can make $u$ as small as we like in the $L_2$ norm on $\Gamma_{x^*}$. Let $\psi^\alpha := u|_{\Gamma_{x^*}}$ and hence $\|\psi^\alpha\|_{L^2(\Gamma_{x^*})} \to 0$ as $\alpha \to 0$. Then $u$ satisfies the following BVP: find $u \in C^2(G^*) \cap C^1(G^* \cup \Gamma_{h^*}) \cap BC(\overline{G^*})$ such that

$$\begin{aligned} u = \psi^\alpha \quad & \text{on } \Gamma_{x^*}, \\ \Delta u + k^2 u = 0 \quad & \text{in } G^*, \\ \frac{\partial u}{\partial x_2} + iku = 0 \quad & \text{on } \Gamma_{h^*}. \end{aligned}$$

**Theorem 3.1** *The above boundary value problem has exactly one solution. Given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$, depending only on $k, f_0, d$, and $\epsilon$, such that, for all $x^* \in \mathbf{R}^2$,*

$$|u(x)| \leq C_\epsilon \|\psi^\alpha\|_{L^2(\Gamma_{x^*})}, \qquad h^* \leq x_2 \leq f_{x^*}(x_1) - \epsilon. \tag{6}$$

Now, denoting the region $G_\epsilon^*$ such that $h^* < x_2 < f_{x^*}(x_1) - \epsilon$, provided $\Gamma$ is contained in $G_\epsilon^*$, we have from Theorem 3.1 that

$$|u(x)| \leq C_\epsilon \|\psi^\alpha\|_{L^2(\Gamma_{x^*})}, \qquad x \in \Gamma, \tag{7}$$

so that $u$ is small on the surface $\Gamma$ if it small on $\Gamma_{x^*}$.

Let

$$w(x) := \int_{\gamma^*} U_1^s(x,y)\phi_{x^*}^\alpha(y)\ \mathrm{ds}(y) - U_1^s(x,x^*), x \in D.$$

Then, recalling the definition of $U_1^s$, in particular that $U_1^s(x,y) = -G_{1,h^*}(x,y)$, $x \in \Gamma$, $y \in D$, it is shown in [5] that $w \in C^2(D) \cap C(\overline{D})$, $\Delta w + k^2 w = 0$ in $D$, $w = -u$ on $\Gamma$, and $w$ satisfies the Sommerfeld radiation conditions (2) and (3). The following continuous dependence result for the above Dirichlet BVP follows from [3].

**Theorem 3.2** *For every $c > 0$ there exists a constant $C > 0$ dependent only on $c$ and $k$, such that, provided $\|f\|_{C^{1,1}(\mathbf{R})} \leq c$, it holds that*

$$|w(x)| \leq C(1 + x_2 - \inf f)^{\frac{1}{2}} \sup_{x\in\Gamma} |u(x)|. \tag{8}$$

Combining the bounds (7) and (8) we have that, for $x \in D$, and provided $\Gamma$ is contained in $G_\epsilon^*$ for some $\epsilon > 0$,

$$|w(x)| \leq CC_\epsilon(1 + x_2 - \inf f)^{\frac{1}{2}} \|\psi^\alpha\|_{L^2(\Gamma_{x^*})} \to 0 \qquad \text{as } \alpha \to 0.$$

In particular this inequality holds for $x = z$. Thus, for $\alpha$ small enough,

$$U_1^s(z,x^*) \approx \int_{\gamma^*} U_1^s(z,y)\phi_{x^*}^\alpha(y)\mathrm{ds}(y). \tag{9}$$

Now, using the reciprocity relation, the definition of $U_1^s(z,y)$ and (9), it follows that

$$G(x^*,z) = G(z,x^*) \approx G_{1,h^*}(z,x^*) + \int_{\gamma^*} (G(y,z) - G_{1,h^*}(z,y))\phi_{x^*}^\alpha(y)\ \mathrm{ds}(y).$$

This last expression is the approximation to the total acoustic field that we compute, i.e. the approximation we use is

$$G^\alpha(x^*,z) := G_{1,h^*}(z,x^*) + \int_{\gamma^*} \left(G(y,z) - G_{1,h^*}(z,y)\right)\phi_{x^*}^\alpha(y)\ \mathrm{ds}(y). \tag{10}$$

**Theorem 3.3** *For every $\epsilon > 0$ and $c > 0$ there exists $C > 0$, dependent only on $k, f_0, d, \epsilon, z_2 - \inf f$, and $c$, such that, provided $\inf(f_{x^*} - f) > \epsilon$, $f_- \leq x_2^* \leq f_+$, and $\|f\|_{C^{1,1}(\mathbf{R})} \leq c$, it holds that*

$$|G(x^*,z) - G^\alpha(x^*,z)| = |w(z)| \leq C\|\psi^\alpha\|_{L^2(\Gamma_{x^*})} = C\|K\phi_{x^*}^\alpha - g_{x^*}\|_{L^2(\Gamma_{x^*})} \to 0 \tag{11}$$

*as $\alpha \to 0$.*

The analysis up to this point neglects the effect of noise in the measured data. In practice we expect to measure $G_\delta(y,z)$ for $y \in \gamma^*$ rather than $G(y,z)$, with $\|G_\delta(\cdot,z) - G(\cdot,z)\|_{L^2(\gamma^*)} = \delta$. Then we compute $G_\delta^\alpha(x^*,z)$, defined by (10) with $G(y,z)$ replaced by the noisy data $G_\delta(y,z)$. From (10) and (11) it follows that

$$|G(x^*,z) - G_\delta^\alpha(x^*,z)| \leq C\|K\phi_{x^*}^\alpha - g_{x^*}\|_{L^2(\Gamma_{x^*})} + \delta\|\phi_{x^*}^\alpha\|_{L^2(\gamma^*)}.$$

From Theorem 3.3 and [4, Theorem 2.24] we see that the first and second terms of this inequality tend, respectively, to zero and infinity as $\alpha \to 0$ with $\delta$ fixed.

# 4 Numerical Results

In this set of numerical experiments we use a pulse of central frequency $f = 1808$ Hz giving a wavelength of $\lambda = 0.19$m. The incident pulse is approximated by a Fourier series of period 5.5ms, using 24 frequencies, ranging from $f = 0$ to $f = 4339$ Hz. The total field is approximated using (10), for 24 equally spaced frequencies from 180 Hz to 4339 MHz, at points $x^*$ above the surface. Denote $G^\alpha(x,z)$ by $G^\alpha_{k_j}$ to indicate its dependance on $k_j$ for $j = 1 \dots 24$. The results are then summed with the appropriate weights, $A_j$, and transformed to the time domain giving the approximation to the total field

$$U^\alpha_N(x,t) := 2\Re \sum_{j=1}^{(N-1)/2} A_j G^\alpha_{k_j}(x,z) e^{-i\omega_j t}, \qquad x \in \mathbb{R}^2 \backslash (\gamma^* \cup \{z\}), t \in \mathbb{R}, \tag{12}$$

for $(N-1)/2 = 24$, (for further details see [4]). For each $x^* = (x_1^*, x_2^*) \in \mathbf{R}^2$, we make the simple choice $f_{x^*}(x) = -\frac{3\lambda}{4} e^{-(\frac{x_1 - x_1^*}{8\lambda})^2} + \frac{\lambda}{4} + x_2^*$, the function defining $\Gamma_{x^*}$. The measurement line, $\gamma^*$, in the inverse problem, has length $|x_1| \le 10\lambda = 1.9$m. In the numerical implementation we approximate the integral in (5) by the trapezium rule with step length $h = \lambda/10$ and collocate at equally spaced points $x$, with the same spacing $h$, only on the part of $\Gamma_{x^*}$ where $|x_1| \le 19\lambda$. This leads to an approximation of (5) as a square linear system, using $\alpha = 10^{-4}$, with coefficient matrix $\mathbf{K}$. We use "measured" values $G(x,z)$ for $x \in \gamma^*$ computed by the boundary integral equation method and super-algebraically convergent Nyström method proposed in [6]. The following figures illustrate the case when $f(x_1) = \frac{11\lambda}{8} + \frac{\lambda}{8}\cos(\frac{2x_1}{\lambda}) + \frac{\lambda}{4}\sin(\frac{2x_1}{3\lambda}), x_1 \in \mathbf{R}$, so that the boundary $\Gamma$ is a non-sinusoidal scattering surface. We fix the height of the measurement line as $H = 4\lambda = 0.76$m and the source of the incident field is at $z = (0, 4\lambda) = (0, 0.76)$m. In Figure 1 we plot the approximation to the total field given by (12) as a function of time (the solid lines) and the actual total field, calculated by the method proposed in [6], (the dotted lines) for $x^* = (0, p\lambda)$, for $p = 0.5, 1.6, 2.6, 3.5$. Clearly, very accurate reconstructions of the field up to the boundary are obtained by this method, from which the position of the boundary should become apparent. In Figures 2 and 3 we plot the reconstructed total field at 8 points in time over one time period for $|x_1| < 5\lambda$ and $1/2\lambda < x_2 < 3/2\lambda$, where $G^\alpha_{k_j}(\cdot, z)$ is calculated at 9.6 points per wavelength in both the $x_1$ and $x_2$ directions. Finally we propose to locate the surface $\Gamma$ as the minimum of

$$P(x) = \frac{1}{T}\int_0^T \Big(U^\alpha_N(x,t)\Big)^2 \mathrm{d}t. \tag{13}$$

By Parseval's theorem,

$$P(x) = 2\sum_{j=1}^{(N-1)/2} |A_j|^2 |G^\alpha_{k_j}(x,z)|^2. \tag{14}$$

Hence in Figure 4 we first interpolate $U^\alpha_N(x,t)$ calculated for $|x_1| \le 5\lambda$ and $1/2\lambda \le x_2 \le 3/2\lambda$ at 9.6 points per wavelength ($\lambda$), to 19 points per wavelength. On the right hand side of the figures we predict the location of the surface by colouring in black the square in which $P(x)$ is minimised as a function of $x_2$. Note that the bottom plots illustrate the case when 5% noise is added to the measurements.

# 5 Conclusions

We have shown that the total acoustic field can be accurately extrapolated up to the boundary without any a priori knowledge of the boundary condition. The location of the boundary can then be construed as the position where the incident and scattered fields merge. For the case when the surface is sound soft we can see that the surface location can be predicted as the minimum of the function (13), (see Figure 4). We see that even when noise is added to the measurement data the extrapolated total acoustic field and the prediction of the surface are still very accurate.

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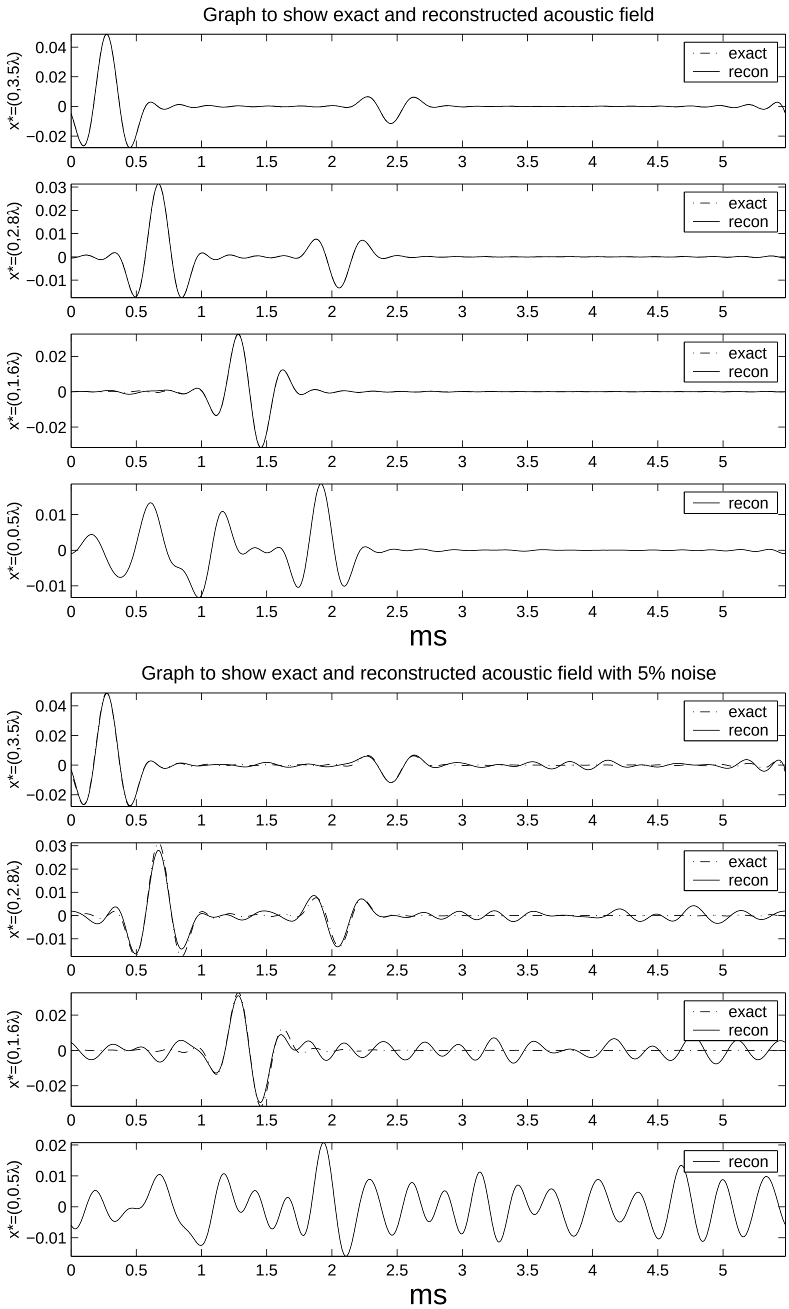


Figure 1: Comparison of the reconstructed total field and the exact total field are shown. The lower figure indicates the case when 5% noise is added to the measurements. The total fields are compared at the points $x = (0, 3.5\lambda)$, $x = (0, 2.6\lambda)$, $x = (0, 1.6\lambda)$ and $x = (0, 0.5\lambda)$ (from top to bottom), for, $0 < t < T$. Note that the surface is at height $1.5\lambda$. Thus the bottom plots show a reconstructed total field for a point $x$ below the surface.

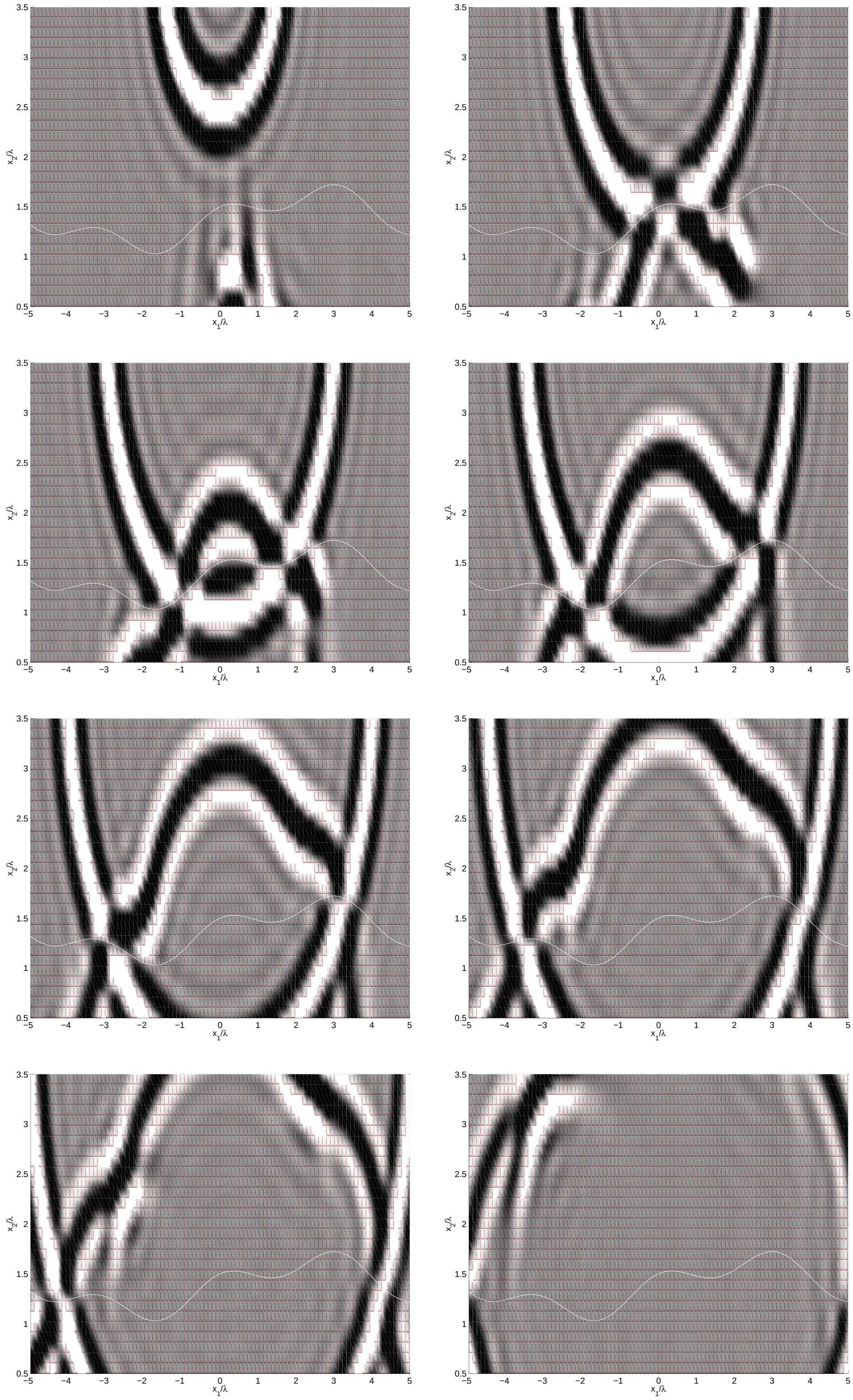

Figure 2: The extrapolated total field for $t = 0.9$ms, 1.4ms, 1.7ms, 2ms, 2.3ms, 2.5ms, 2.8ms and 3.4ms (from top left to bottom right).

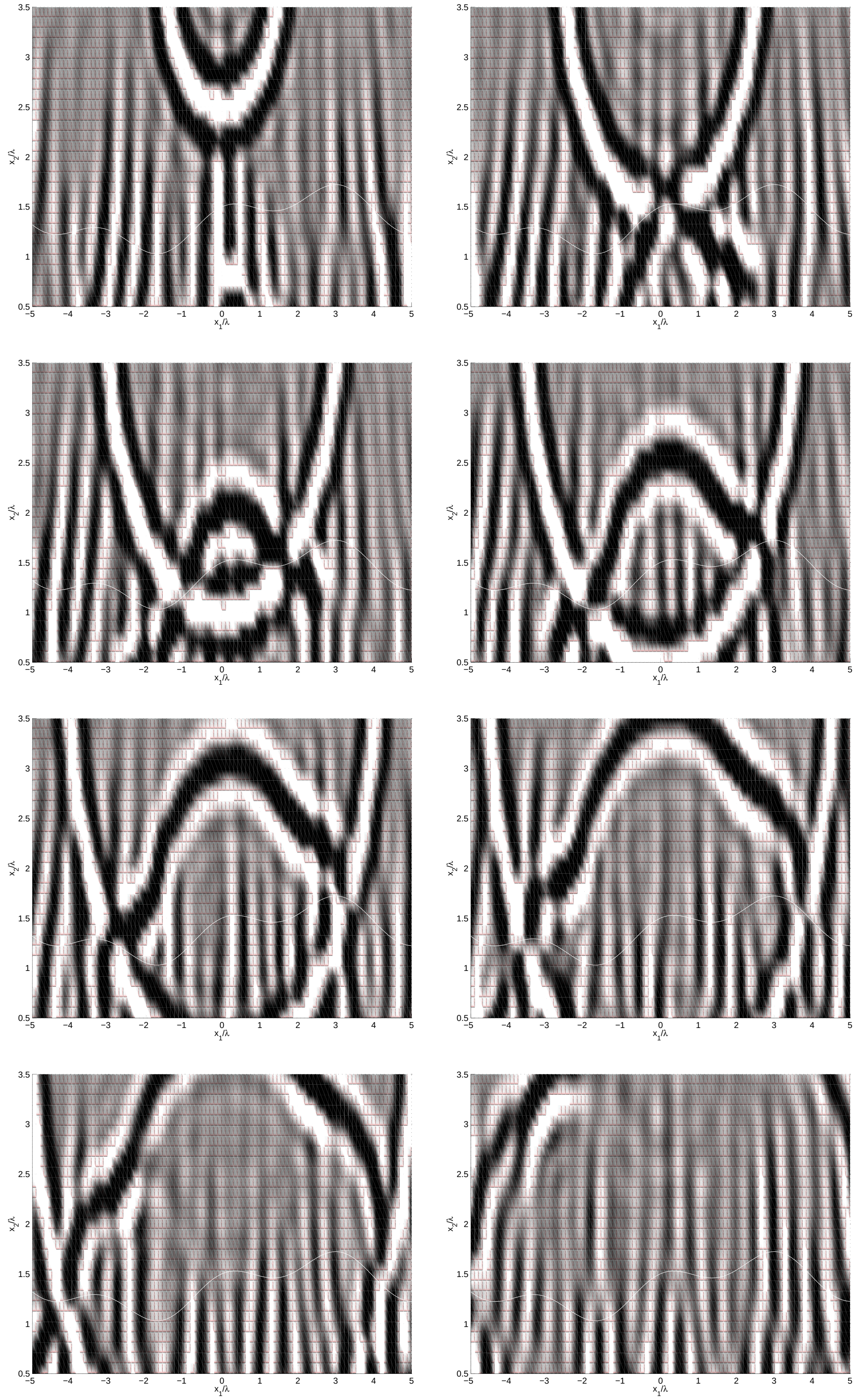


Figure 3: The extrapolated total field for $t = 0.9$ms, 1.4ms, 1.7ms, 2ms, 2.3ms, 2.5ms, 2.8ms and 3.4ms (from top left to bottom right), where 5% noise added to the measurement data.

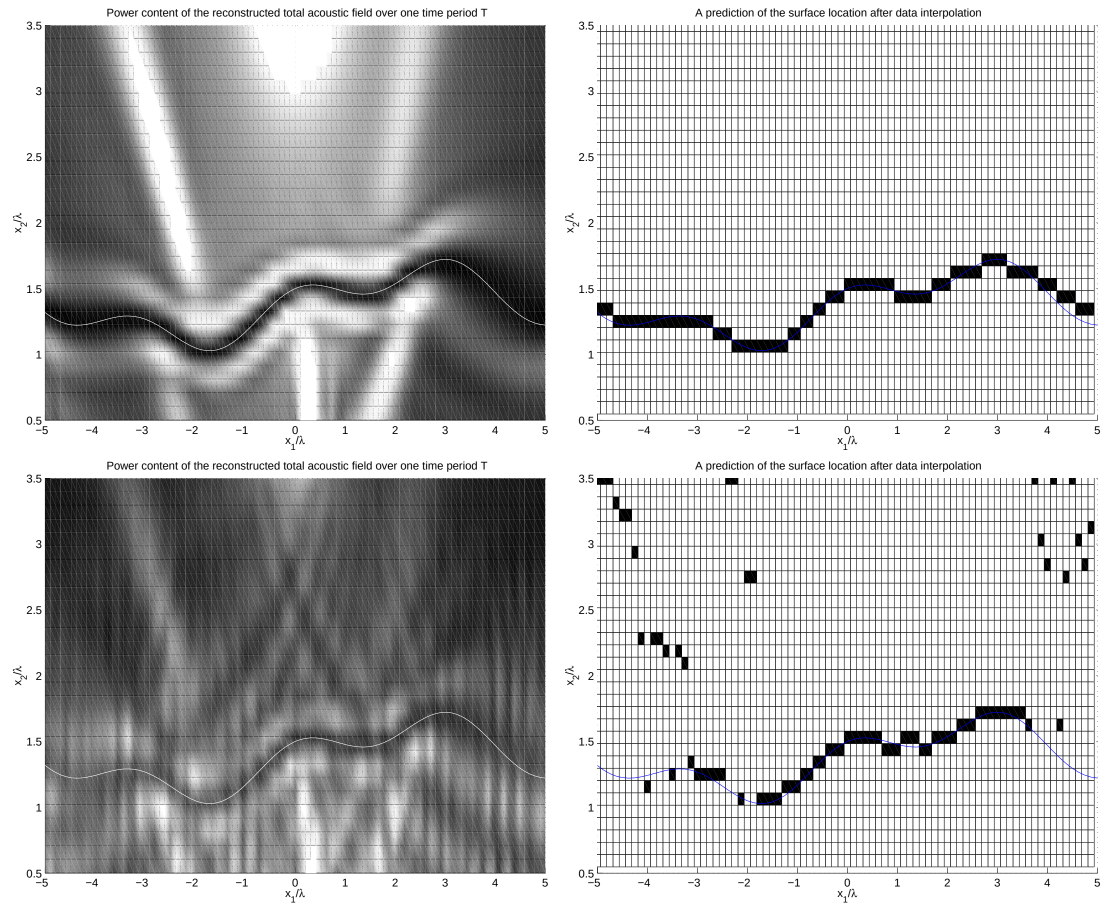


Figure 4: A plot of the power content, $P(x)$, of the acoustic field (left hand side). On the right hand side we predict the surface location from this plot by colouring in, in each column, the square in which $P(x)$ is minimised as a function of $x_2$. The bottom two plots show the case when 5% noise is added to the measurements.