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FREE VIBRATION ANALYSIS OF A CLASSICAL H-SHAPED FRAME BY A THREE-DIMENSIONAL STRUCTURAL MODEL

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In this paper, vibrations in an H-shaped planar frame are analyzed from a wave vibration standpoint based on a three-dimensional (3D) structural model. In the wave vibration approach, vibrations are described as waves that propagate along uniform structural elements and are reflected and transmitted at structural discontinuities such as joints and boundaries. All possible vibration motions, namely, the in-plane coupled bending and axial vibrations and the out-of-plane coupled bending and torsional vibrations, are studied based on classical vibration theories. The 3D theoretical analysis results are compared with existing results obtained using two-dimensional (2D) structural model. The 3D structural model based wave technique is seen to provide a concise and accurate single step process in analyzing in- and out-of-plane vibrations in H-shaped planar frames.

Keywords: wave vibration, H frame, 2D and 3D model, Euler-Bernoulli beam theory

1. Introduction

A wave based vibration analysis approach is applied in studying vibrations in an H-shaped planar frame. From the wave standpoint, vibrations propagate along a uniform waveguide (that is, a structural element), and are reflected and transmitted at discontinuities (such as joints and boundaries) [1-4]. The propagation relations are governed by the equations of motion, while the reflection and transmission relations are determined by the equilibrium and continuity conditions at the local discontinuities. It is found that when a 3D structural model is adopted, the wave vibration approach provides a one step process in analysing in- and out- of plane vibrations in planar rectangular frames. Free and forced vibrations of the coupled axial and in-plane bending vibrations, as well as the coupled torsional and out-of-plane bending vibrations are obtained accurately [5].

In this study, the 3D structural model based wave vibration approach is extended to analyse free vibrations in a planar H shaped frame based on classical Euler-Bernoulli vibration theory. Equilibrium and continuity conditions at the T-shaped structural discontinuity are obtained based on a 3D structural analysis. This paper is organized as follows. In the next section, vibration waves in the H frame are assembled through propagation, reflection, and transmission relations. Section 3 describes the establishment of equilibrium and continuity conditions. In section 4, wave analysis results of an example H frame are presented. Good agreement has been reached with existing studies on in-plane vibration analysis of the same frame. Conclusions are drawn in section 5.

2. Wave vibration analysis

Figure 1 shows a planar H-shaped frame. It consists of five uniform beam elements AC, CE, BD, DF, and CD. Consequently there exist five pairs of propagation relations. At the boundaries A, B, C, and D of the planar frame, there exist four reflection relations. There are two T-shaped joints, each contains three equations in describing the relations among the incoming and outgoing waves from the beam elements joined at the joint. Details are as follows [6, 7]:

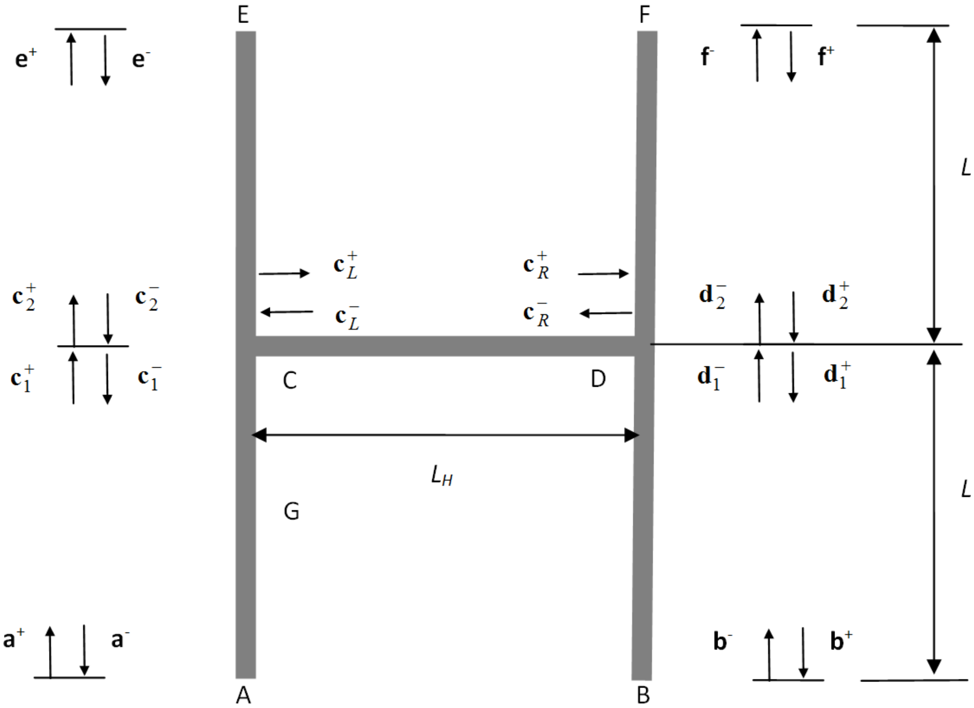


Figure 1: Wave Vibrations of an H-Shaped Frame

- Propagation relations along uniform beam elements

$$\mathbf{c}_1^+ = \mathbf{f}(L)\mathbf{a}^+ ,\ \mathbf{a}^- = \mathbf{f}(L)\mathbf{c}_1^- ; \mathbf{e}^+ = \mathbf{f}(L)\mathbf{c}_2^+ ,\ \mathbf{c}_2^- = \mathbf{f}(L)\mathbf{e}^- ; \mathbf{d}_1^- = \mathbf{f}(L)\mathbf{b}^- ,\ \mathbf{b}^+ = \mathbf{f}(L)\mathbf{d}_1^+ ;$$
$$\mathbf{d}_2^+ = \mathbf{f}(L)\mathbf{f}^+ ,\ \mathbf{f}^- = \mathbf{f}(L)\mathbf{d}_2^- ; \mathbf{c}_R^+ = \mathbf{f}(L_H)\mathbf{c}_L^+ ,\ \mathbf{c}_L^- = \mathbf{f}(L_H)\mathbf{c}_R^- ; \quad (1)$$

- Reflection and transmission relations at T joint C, where Beam CE, CD, and CA are named Beam 1, 2, and 3, respectively

$$\mathbf{c}_2^+ = \mathbf{r}_{11}\mathbf{c}_2^- + \mathbf{t}_{31}\mathbf{c}_1^+ + \mathbf{t}_{21}\mathbf{c}_L^- ,\ \mathbf{c}_1^- = \mathbf{r}_{33}\mathbf{c}_1^+ + \mathbf{t}_{13}\mathbf{c}_2^- + \mathbf{t}_{23}\mathbf{c}_L^- , \mathbf{c}_L^+ = \mathbf{r}_{22}\mathbf{c}_L^- + \mathbf{t}_{12}\mathbf{c}_2^- + \mathbf{t}_{32}\mathbf{c}_1^+ ; \quad (2)$$

- Reflection and transmission relations at T joint D, where Beam DB, DC, and DF are named Beam 1, 2, and 3, respectively

$$\mathbf{d}_1^+ = \mathbf{r}_{11}\mathbf{d}_1^- + \mathbf{t}_{31}\mathbf{d}_2^+ + \mathbf{t}_{21}\mathbf{c}_R^+ , \mathbf{d}_2^- = \mathbf{r}_{33}\mathbf{d}_2^+ + \mathbf{t}_{13}\mathbf{d}_1^- + \mathbf{t}_{23}\mathbf{c}_R^+ , \mathbf{c}_R^- = \mathbf{r}_{22}\mathbf{c}_R^+ + \mathbf{t}_{12}\mathbf{d}_1^- + \mathbf{t}_{32}\mathbf{d}_2^+ ; \quad (3)$$

- Reflections at boundaries

$$\mathbf{a}^+ = \mathbf{r}_A\mathbf{a}^- ;\ \mathbf{b}^- = \mathbf{r}_B\mathbf{b}^+ ;\ \mathbf{e}^- = \mathbf{r}_E\mathbf{e}^+ ;\ \mathbf{f}^+ = \mathbf{r}_F\mathbf{f}^- ; \quad (4)$$

where **f**, **r**, and **t** denote the propagation, reflection, and transmission matrices, respectively. The letter subscripts in the reflection matrices refer to the corresponding boundary, and the number subscripts in the reflection and transmission matrices refer to the related beam elements. The reflection and transmission matrices due to wave incident from beam *i* are denoted as $\mathbf{r}_{ii}$, $\mathbf{t}_{ij}$, and $\mathbf{t}_{ik}$, where *j* and *k* refer to the remaining two beam elements joined at the same T-joint.

The above equations can be assembled in matrix form as

$$\mathbf{Az} = \mathbf{0} ; \quad (5)$$

where **A** is a 120 by 120 square coefficient matrix and **z** is a wave component vector of size 120. This is because there are 20 wave components each containing six elements (two each for in- and out- of plane bending vibrations and one each for axial and torsional vibrations). Setting the determinant of the coefficient matrix **A** to zero gives the natural frequencies of the H-shaped frame.

3. Equilibrium and continuity conditions at a T-shaped joint based on 3D model

The equations of motion for in- and out- of plane bending, axial, and torsional vibrations of a uniform beam element lying along *x*-axis by classical vibration theories are [8]

$$\begin{aligned} EI_z \frac{\partial^4 y(x,t)}{\partial x^4} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} &= q_1(x,t); \\ EI_y \frac{\partial^4 z(x,t)}{\partial x^4} + \rho A \frac{\partial^2 z(x,t)}{\partial t^2} &= q_2(x,t); \\ \rho A \frac{\partial^2 u(x,t)}{\partial t^2} - EA \frac{\partial^2 u(x,t)}{\partial x^2} &= p(x,t); \\ \rho J_x \frac{\partial^2 \theta(x,t)}{\partial t^2} - GJ_x \frac{\partial^2 \theta(x,t)}{\partial x^2} &= r(x,t); \end{aligned} \tag{6}$$

where x is the position along the beam axis, t is the time, $y(x,t)$, $z(x,t)$, $u(x,t)$, and $\theta(x,t)$ are the *xy*-plane, *xz*-plane bending, axial, and torsional deflections of the centerline of the beam, respectively. $q_1(x,t)$, $q_2(x,t)$, $p(x,t)$ and $r(x,t)$ are the externally applied bending forces (*xy*-plane and *xz*-plane), axial force, and torque, respectively. A is the cross-sectional area, I_z and I_y are the area moment of inertia of cross-section about the *z*-axis and the *x*-axis, respectively. J_x denotes the polar moment of area of the cross-section about the *x*-axis. ρ, E, and G are the volume mass density, Young's modulus, and shear modulus, respectively.

The in- and out-of-plane shear forces $V_y(x,t)$ and $V_z(x,t)$, in- and out-of-plane bending moments $M_y(x,t)$ and $M_z(x,t)$, axial force $F(x,t)$ and torque $T(x,t)$ at a cross section of the beam are related to the in- and out-of-plane bending deflections $y(x,t)$ and $z(x,t)$, the in- and out-of-plane bending slopes $\varphi_y(x,t)$ and $\varphi_z(x,t)$, the axial deflection $u(x,t)$, and the torsional deflection $\theta(x,t)$ by

$$\begin{aligned} V_y(x,t) &= -EI_y \frac{\partial^3 y(x,t)}{\partial x^3},\ V_z(x,t) = -EI_z \frac{\partial^3 z(x,t)}{\partial x^3}, \\ M_y(x,t) &= EI_y \frac{\partial \varphi_y(x,t)}{\partial x},\ M_z(x,t) = EI_z \frac{\partial \varphi_z(x,t)}{\partial x}, \\ F(x,t) &= EA \frac{\partial u(x,t)}{\partial x},\ T(x,t) = GJ_x \frac{\partial \theta(x,t)}{\partial x}, \end{aligned} \tag{7}$$

where $\varphi_y(x,t) = \frac{\partial y(x,t)}{\partial x}$ and $\varphi_z(x,t) = \frac{\partial z(x,t)}{\partial x}$ according to the classical Euler-Bernoulli theory. The positive signs for shear forces, bending moments, axial force, and torque are defined in Figure 2.

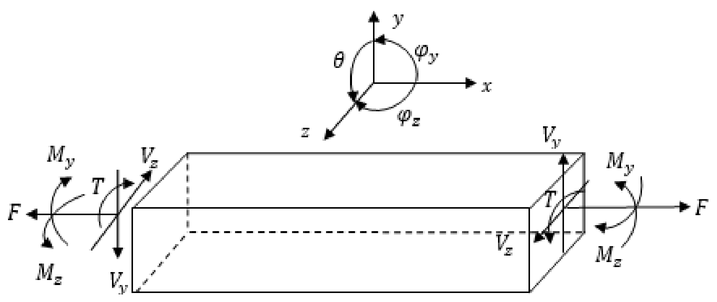


Figure 2: Definition of positive signs for shear forces, axial force, bending moments, and torque

The equilibrium and continuity conditions are obtained from the free body diagram shown in Figure 3 as follows:

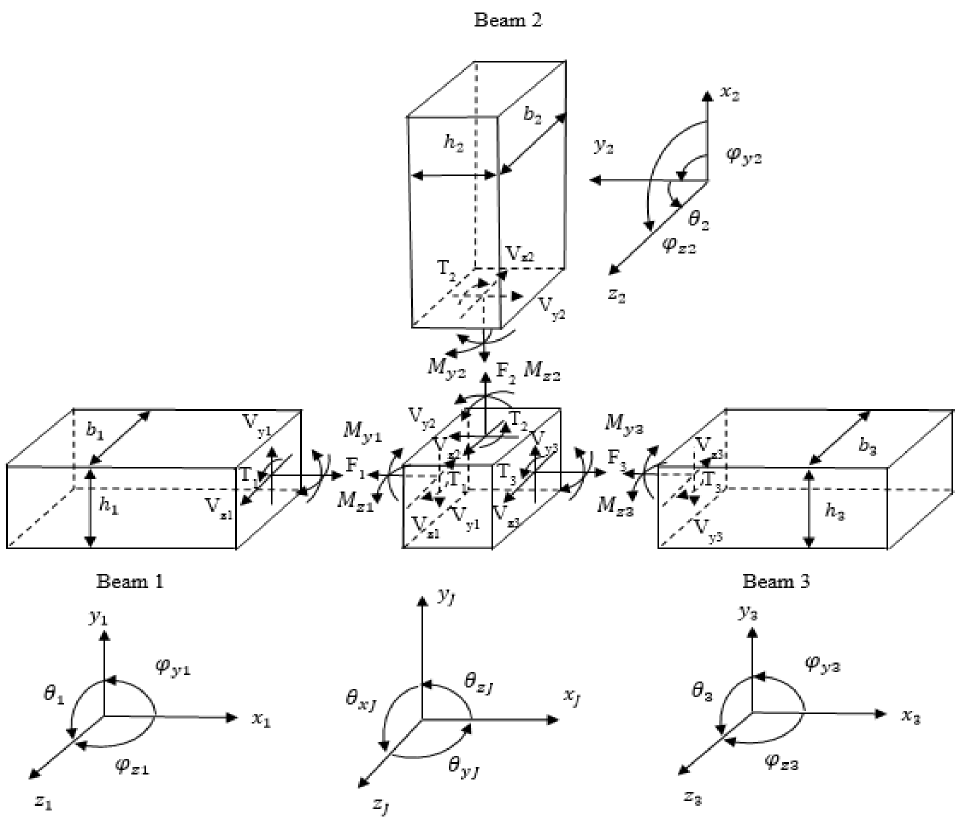


Figure 3: Free body diagram of a T joint based on 3D model

$$
\begin{gathered}
-V_{y2} - F_1 + F_3 = m\ddot{x}_J, \\
F_2 - V_{y1} + V_{y3} = m\ddot{y}_J, \\
M_{y2} - M_{y1} + M_{y3} + V_{y2}\frac{h_1}{2} + V_{y1}\frac{h_2}{2} + V_{y3}\frac{h_2}{2} = J_{zJ}\ddot{\theta}_{zJ}, \\
V_{z2} - V_{z1} + V_{z3} = m\ddot{z}_J, \\
-T_1 + M_{z2} + T_3 + V_{z2}\frac{h_1}{2} = J_{xJ}\ddot{\theta}_{xJ}, \\
T_2 + M_{z1} - M_{z3} - V_{z1}\frac{h_2}{2} - V_{z3}\frac{h_2}{2} = J_{yJ}\ddot{\theta}_{yJ};
\end{gathered}
\tag{8}
$$

$$
\begin{gathered}
x_J = u_1,\ y_J = u_2,\ x_J = u_3, \\
y_J = y_1 + \theta_{zJ}\frac{h_2}{2},\ x_J = -y_2 + \theta_{zJ}\frac{h_1}{2},\ y_J = y_3 - \theta_{zJ}\frac{h_2}{2}, \\
z_J = z_1 - \theta_{yJ}\frac{h_2}{2},\ z_J = z_2 - \theta_{xJ}\frac{h_1}{2},\ z_J = z_3 + \theta_{yJ}\frac{h_2}{2}, \\
\theta_{xJ} = \theta_1 = \theta_3 = \varphi_{z2},\ \theta_{yJ} = \theta_2 = -\varphi_{z1} = -\varphi_{z3},\ \theta_{zJ} = \varphi_{y1} = \varphi_{y2} = \varphi_{y3};
\end{gathered}
\tag{9}
$$

where subscript *J* refers to the joint, *x*, *y*, *z*, and 1, 2, 3 denote the corresponding axes and beam elements, respectively.

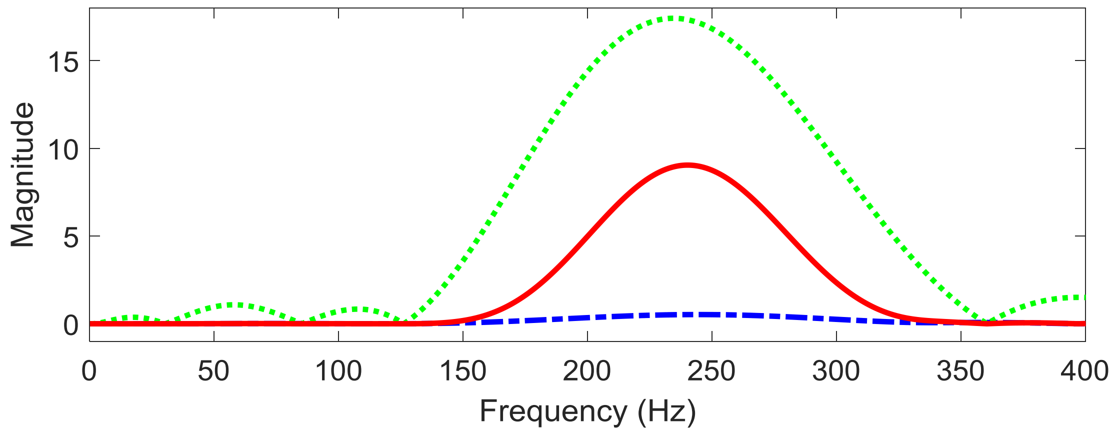
The reflection and transmission relations are obtained from the above equilibrium and continuity conditions by following a similar procedure as described by the author in [6, 7]. Note that there are three sets of reflection and transmission relations, corresponding to waves incident from Beam 1, 2, and 3, respectively.

4. Wave analysis results of an example H frame

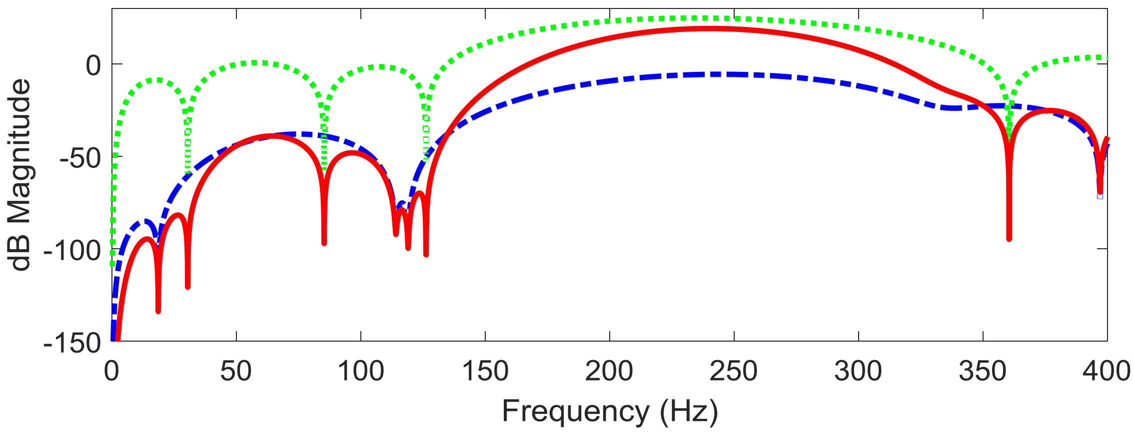
In-plane vibrations in H frames have been studied previous [6, 7, 9]. For comparison purposes, the same frame with the following physical properties are chosen as a numerical example: Young's modulus $E = 206\,GN/m^2$, mass density $\rho = 7800\,mg/m^3$, the cross section of the beam elements is $1.27\times10^{-2}m$ by $1.27\times10^{-2}m$, and the lengths of the uniform beam elements between any two discontinuities are $0.3m$, that is, $L = L_H = 0.3m$. Boundaries A and B are simply supported, and E and F are free. For comparison purposes, the natural frequencies corresponding to in-plane vibrations obtained by Lee and Ng [9] and Mei [6, 7] are presented in Table 1.

Figure 4 shows the linear and dB magnitude plots of the characteristic polynomial of the H frame corresponding to in-plane, out-of-plane, and three dimensional vibration analysis. The *x* values of the zeros on the linear magnitude plot and of the local minima on the dB magnitude plot correspond to the natural frequencies. The natural frequencies are recorded in Table 2. By comparing to results in Table 1, it can be seen that good agreement has been reached in natural frequencies of in-plane vibrations.

It shall be mentioned that since the cross section of the example frame is rotationally asymmetric, the torsional rigidity is adjusted by an adjustment coefficient $\beta = 0.141$ corresponding to the 1:1 side ratio of the square cross section according to Cremer et. al. [2]



(a)



(b)

Figure 4: Linear (a) and dB (b) magnitude plot of the characteristic polynomial corresponding to in-plane (…), out-of-plane (-.-.-.), and three dimensional vibration analysis

Table 1: Natural frequencies of In Plane Vibration

Sequence	Natural frequencies				
	$f_n\ (Hz)$		Non-dimensional $\Omega = 2\pi f_n L^2 \sqrt{\frac{\rho A}{EI}}$		
	Mei [6]	Mei [7]	Mei [6]	Mei [7]	Lee and Ng [9]
1	30.4	30.4	0.9	0.9	0.92059
2	85.2	85.1	2.6	2.6	2.58654
3	126.2	126.0	3.8	3.8	3.82212

Table 2: Natural frequencies from Two- and Three- Dimensional Analysis

Sequence	Natural frequencies $f_n\ (Hz)$		
	2D In Plane	2D Out of Plane	3D Analysis
1		18.6	18.6
2	30.4		30.4
3	85.2		85.2
4		113.3	114.2
5		118.8	119.2
6	126.2		126.2
7	360.4		360.4
8		397.5	397.0

5. Conclusions

A three-dimensional structural model is proposed in analyzing vibrations in a planar H frame by applying the wave vibration analysis technique. The 3D analysis can predict all possible vibration motions, namely, the in-plane coupled bending and axial vibrations and the out-of-plane coupled bending and torsional vibrations, through a one step process. The wave based vibration analysis approach is systematic. The accuracy of the approach is confirmed by comparison to available analysis results of in-plane vibrations of the same H shaped frame.

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