MEAN INTENSITY AND VERTICAL MUTUAL FUNCTION OF ACOUSTIC SIGNAL IN A RANDOM MEDIUM: THE OCEAN. STOCHASTIC AND STATISTICAL APPROACHES.

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1. INTRODUCTION

In underwater acoustics, an important problem is to localize and identify a source from acoustic signals received on an antenna. But the propagation of acoustic waves in the ocean cannot always be considered as fully deterministic because of the influence of oceanic fluctuations on the acoustic field. The aim of this paper is to characterize the effects of these fluctuations on sound propagation, and to present some numerical results.
Propagation in a deterministic medium is now well known and several methods to investigate the sound field in such a case are available (geometric rays, normal modes, parabolic approximation). These methods consider deterministic sound speed profiles and very simple characteristics for the boundaries (surface and bottom). But the real environment is more complex and waves are scattered by fluctuations of the medium or surface inhomogeneities, and reflection on a complex bottom. This leads to stochastic changes in the amplitude and phase of acoustic signals. The sound field is then considered as a random function, characterized by its moments: intensity, vertical coherence, variance of intensity fluctuations...
The problem studied here is two-dimensional.

Environmental phenomena are first reviewed in order to characterise the random data, then Itô's formalism is used to generalize the derivation of parabolic equations for all the moments of the sound field. The equation of the second-order moment is solved numerically. The numerical results obtained are compared with those obtained by statistical method.

2- MEDIUM FLUCTUATIONS

Environmental phenomena are presented in the following table in decreasing order of importance [1]:

Oceanic phenomena	Spatial scales H:horizontal V: vertical	Time scales	Magnitude of sound speed variations
Ocean climate (general circulation)	H: oceanic basin V: few 100m	seasonal	10 m/s
Mesoscale (Meteorology)	H: 50 to 500 km V: ocean deep	days → month	1m/s
Internal waves	H: 100m to 10 km V: 1 to 100 m	20mn → 1 day	0,1m/s
Fine structures	H: few 10m V: 1 to 10m	< 20mn	< 0,01m/s
Microstructures	H: few meters V: centimeters		
Tidal	H: variable V: <10m	diurne semidiurne	

The first two phenomena which have the most important effects are generally handled by deterministic modelling. Fluctuations of sound speed resulting from the effects of internal waves constitute prevailing disturbing effects.
Fluctuations in the medium due to internal waves are taken into account in the index of refraction:

r: horizontal propagation axis, z: vertical axis

n : index of refraction defined by: $n(r,z) = \dfrac{C_0}{C(r,z)}$

$C(r,z) = C(z) + \delta C(r,z)$

$C(z)$: deterministic sound velocity, C_0: reference sound speed, $\delta C(r,z)$: fluctuations of velocity

The square of the index of refraction is written as a sum of a deterministic part n_0^2 and a random part ε:

$$n^2(r,z) = n_0^2(z) + \varepsilon(r,z).$$

$$\text{with} \quad \varepsilon(r,z) \approx \frac{2\delta C(r,z)}{C_o}$$

$\varepsilon(r,z)$ is supposed to be gaussian. It is determined by its first two moments, chosen as:
- mean sound speed fluctuations: $< \varepsilon(r,z) > = 0$
- autocorrelation function of sound speed:

$$< \varepsilon(r,z)\, \varepsilon(r',z') > = \varepsilon_o^2\left(\frac{z+z'}{2}\right) \exp\left[\frac{-(r-r')^2}{2\, L_h^2\left(\frac{z+z'}{2}\right)}\right] \exp\left[\frac{-(z-z')^2}{2\, L_v^2\left(\frac{z+z'}{2}\right)}\right]$$

When the propagation range becomes very large (≈100km) compared with the horizontal spatial correlation lengh (≈ 2 km), the assumption of delta-correlation is reasonable and the gaussian function $\left[\frac{1}{\sqrt{2\pi}L_h} e^{\frac{-(r-r')^2}{2L_h^2}} \right]$ is approached by a Dirac function $\delta(r-r')$. The autocorrelation function is then:

$$<\varepsilon(r,z)\varepsilon(r',z')> = \delta(r-r')\ A(z-z', \frac{z+z'}{2})$$

with $A\ (z-z', \frac{z+z'}{2}) = \sqrt{2\pi}L_h\ \varepsilon_0^2(\frac{z+z'}{2})\ \exp\left[\frac{-(z-z')^2}{2\ L_v^2\,(\frac{z+z'}{2})} \right]$

This simple analytic function can be introduced in propagation model more easily than the more realistic, but more complex too, spectrum of internal waves (Fourier transform of sound speed autocorrelation function) established by Garrett and Munk from in-situ measurements [2].

3 - STATISTICAL APPROACH

In this paragraph, the moments of sound pressure are obtained by a Monte-Carlo approach.

The sound pressure P(r,z) in the ocean satisfies the Helmholtz equation:

$$\Delta P(r,z) + k_0^2\, n^2(r,z)\, P(r,z) = \delta(z-z_s)\ \delta(\frac{1}{2\pi r})$$

We also assume a free surface condition at the surface (z=0), a Neumann condition on the bottom (z=H) and conditions at infinity.

Let the sound field be written u with: $P(x,z) = \frac{e^{ik_0 x}}{\sqrt{x}}\ u(x,z)$

The parabolic approximation is still valid when the fluctuations are caused by internal waves [2]. For a point far from source, u is the solution of the Standard Parabolic Equation (SPE):

$$2ik_0 \frac{\partial(u)}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k_0^2 \left(n_0^2(z) + \varepsilon(r,z) - 1\right)u(r,z) = 0 \qquad (\text{S.P.E})$$

This equation is then solved numerically by a clasical finite-differences method for any profile $\varepsilon(r,z)$.

A program have been used to generate random sequences of ε, from the expression of the autocorrelation function and its Fourier transform (autospectrum) [3] [4].

For each sequence of $\varepsilon(r,z)$, the solution is $u(r,z)$ is computed and stored. The statistical moments of the sound field are then obtained by summing and averaging the results obtained for a large number of sequences. In particular, the mean sound level is calculated by: $PP(r,z) = -10 \log |<u(r,z)>|^2 + 10 \log |r|$
and the mean intensity level by:

$$PP(r,z) = -10 \log <|u(r,z)|^2> + 10 \log |r|$$

Figure (1) presents some numerical results for the second-order moment. The drawback of this method results in that it requires to solve the parabolic equation for a large number of realization ε; it is then highly time-consuming. It is then very interesting to develop equations satisfied by the successive moments of the sound field.

4 - PARABOLIC EQUATIONS FOR THE MOMENTS

Taken ensemble average of the (S.P.E) leads to:

$$2ik_0 \frac{\partial <u(r,z)>}{\partial r} + \frac{\partial^2 <u(r,z)>}{\partial z^2} + k_0^2 (n_0^2(z) - 1)<u(r,z)> + k_0^2 <\varepsilon(r,z)\, u(r,z)> = 0$$

that points out a coupled term in ε and u. Tatarskii has solved the problem under the assumption of white gaussian noise for the index of refraction. Itô's formalism is used here to generalize the derivation of parabolic equations for all the moments of the sound field under the same hypotheses. To present simple calculations, we first use a semi-discretized versus of the (S.P.E.) in the z-direction:

$$\frac{d}{dr}\begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_m \\ . \\ u_M \end{pmatrix} = \begin{pmatrix} d_1 & b & & 0 \\ b & d_2 & & . \\ 0 & b & & . \\ & & \cdots & . \\ . & . & & . \\ . & . & & 0 \\ . & . & & b \\ 0 & 0 & & d_M \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_m \\ . \\ u_M \end{pmatrix} + \begin{pmatrix} u_1 & 0 & & 0 \\ 0 & u_2 & & . \\ . & 0 & & . \\ . & . & \cdots & . \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & & u_M \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ . \\ . \\ . \\ \sigma_m \\ . \\ \sigma_M \end{pmatrix}$$

$$d\underline{B}(r) = \underline{\sigma} = \frac{ik_0}{2}\underline{\varepsilon}(r) \qquad \underline{\varepsilon}(r) = (\varepsilon_1(r), \varepsilon_2(r), \ldots, \varepsilon_M(r))$$

$$\boxed{d\underline{u}(r) = \underline{f}(\underline{u}(r))dr + G(\underline{u}(r)).\, d\underline{B}(r)dr \qquad , r > 0 \text{ (S.D.E.)}}$$

with $u(0) = u0$

$d\underline{B}$: is proportional to $\underline{\varepsilon}(r)$, so it is a gaussian white noise, and this equation is a stochastic differential equation of the Itô's form. Formally, $\underline{u}(r)$ is solution of the integral equation:

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$$\underline{u}(r) = \underline{u}_0 + \int_0^r \underline{f}(\underline{u}(s))\, ds + \int_0^r G(\underline{u}(s))\, d\underline{B}(s)$$

The peculiar point is then the definition of the stochastic integral [5]. The choice of Itô's definition implies mathematical solutions where u and ε are decorrelated

(i.e $< \int_{r_1}^{r_2} G(\underline{u}(s))\, d\underline{B}(s) > = 0 \quad \forall\ 0 \leq r_1 < r$). Stratonovitch's choice implies physical solutions, and consist in adding a corrective term which is simple in this cases and the equation (S.D.E.) can be written as:

$$du_m(r) = -\frac{k_0^2}{8} A(0,z_m) u_m dr \quad + \quad f_m(\underline{u}(r))dr + \sum_j G_{mj} dB_j(r) \qquad \text{(E1)}$$

corrective term of Stratonovitch — Classical calculus of Itô

This is still a new Itô's equation for which we can use Itô's stochastic calculus to obtain equations for the moments. For example to determine the equation of the first moment, we have to evaluate the quantity:

$$< u_i > = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} u_i\, p(\underline{u}(r))\ du_1 \dots du_M$$

Using Itô's formula, we obtain a discrete versus of the Tatarskii's equation:

$$2ik_0 \frac{\partial <u>}{\partial r} + \frac{\partial^2 <u>}{\partial z^2} + k_0^2 (n_0^2(z) - 1) <u> + \frac{ik_0^3}{4} A(0,z) <u> = 0 \qquad \text{(M.P.E)}$$

This formalism allows one to obtain by the same way equation of all the moments.
Further more, the Itô's formalism is well adapted for relaxing the hypotheses of gaussian white noise: the application of the Papanicolaou-Kohler theorem allows one to extend the validity of the numerical results for the more realistic hypothesis on a gaussian large band noise (with finite energy) for various estimated ranges of propagation [6].
$\varepsilon(r,z)$ is modelised by $\varepsilon^\nu(r,z)$ which becomes a gaussian white noise when ν tends to 0

$$d\underline{u}(r) = \underline{f}(\underline{u}(r))\, dr + \underline{G}(\underline{u}(r)).\varepsilon^\nu(r) dr$$

Papanicolaou-Kohler theorem's conclusion is that: when ν tends to 0, $\underline{u}$ tends to $\underline{u}\#$ which is solution of (E1).
We make then the following choice for ε^ν:

$$\langle \varepsilon^\nu(r,z) \varepsilon^\nu(r',z') \rangle = \frac{1}{\nu^2} \langle \varepsilon(\frac{r}{\nu^2},z) \varepsilon(\frac{r'}{\nu^2},z') \rangle = \varepsilon_0^2(z) \sqrt{2\pi L_h}\ \frac{1}{\nu^2 \sqrt{2\pi L_h}} e^{\frac{-(r-r')^2}{2L_h^2 \nu^4}}\ e^{\frac{-(z-z')^2}{2L_v^2}}$$

Variance is: $\sigma = \varepsilon_0^2(z)\ \sqrt{2\pi}\, L_h\ \sqrt{2\pi}\, L_v.$

after variable changement $r \longrightarrow \frac{r}{v^2}$ S.P.E becomes:

$$\frac{\partial u}{\partial r} = \frac{i}{2}\left[\frac{1}{k_0}\frac{\partial^2}{\partial z^2} + k_0(n_0^2(z)-1)\right] \frac{u}{v^2} + \frac{ik_0}{2}\sigma\, \varepsilon^v\left(\frac{r}{v^2}, z\right)\frac{1}{v}$$

where v is a little parameter whose unity is $m^{-1/2}$. The second result of the theorem is that: if the first term of the right side is of the order of one, and the second of one divided by v, then the parabolic equation of the first moment in case of white noise is a good approximation for propagation ranges of order of one by v^2.
The next table gives some numerical examples:

k_v^2	10^{-6}	10^{-4}	$4\ 10^{-6}$
L_h	50 000m	2 000m	10 000m
L_v	1 000m	100m	500m
ε_0	$5\ 10^{-4}$	10^{-3}	$5\ 10^{-4}$
F	100Hz	500Hz	300Hz
D	850km	40km	200km

The parabolic equation of the second-order moment is solved numerically by modal expansion leading to the mean intensity, the vertical mutual coherence, and by taking the spatial Fourier transform of the vertical mutual coherence function, to the angular spectral density (figure (2)). This method takes phase into account, it is then possible to observe the coherent effects [4].

5 - CONCLUSIONS

Two methods have been used to obtain the moments of first and second order. The stochastic method is based on the Ito's formalism, leading to equations for all the moments of the sound field. These equations are of course those previously obtained by other means but the interest of this calculus is that it can take into account the more realistic hypothesis of a gaussian large band noise (with finite energy). Further more the Papanicolaou-Kohler theorem provides an estimation of the validity domain of the equations.
The numerical results of the both methods have been compared. They have also been compared with those obtained with experimental profiles (in-situ measurements). All the results are in good agreement. Fluctuations on the sound intensity level can go up to 8dB.
The statistical method must be further developed, although it is time-consuming. It is still a convenient way to calculate the fourth-order moment (the stochastic equation is

quite difficult to solve) and it must be noted that the computing time and the storage size are not very highly increased between the computations of the second-order and the fourth-order moments.

6 - REFERENCES

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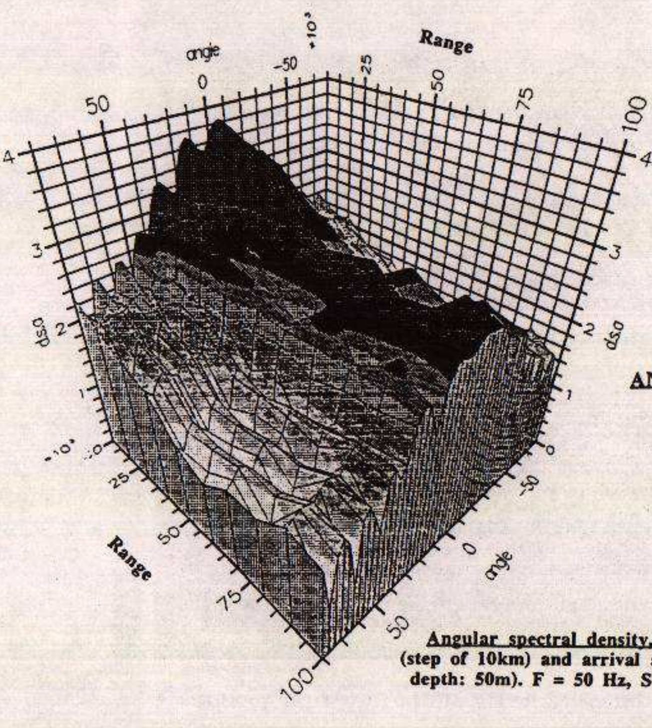


Figure (2):

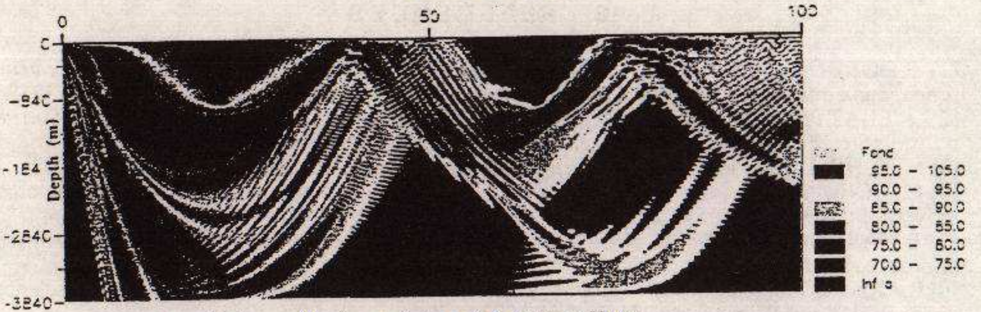
ANGULAR SPECTRAL DENSITY

Angular spectral density, function of propagation range (step of 10km) and arrival angle on an antenna (length:200m, depth: 50m). F = 50 Hz, Source: Hs= 50m, Bottom = 800m.

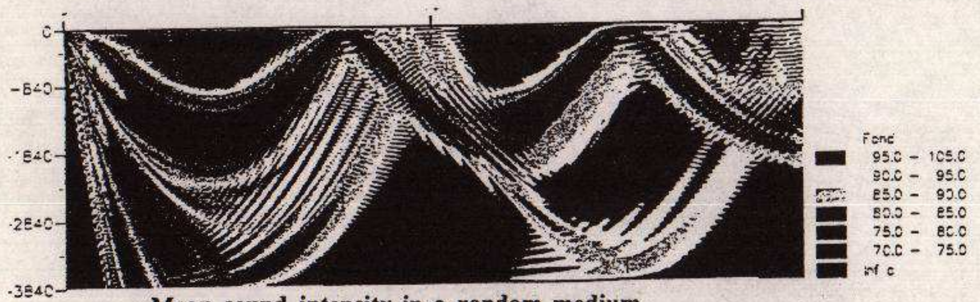
Figure (1):

Intensity levels determined by a statistical approach.
Random sound speed profiles have been numerically generated from specific characteristics of fluctuations, calculated from in-situ measurements.

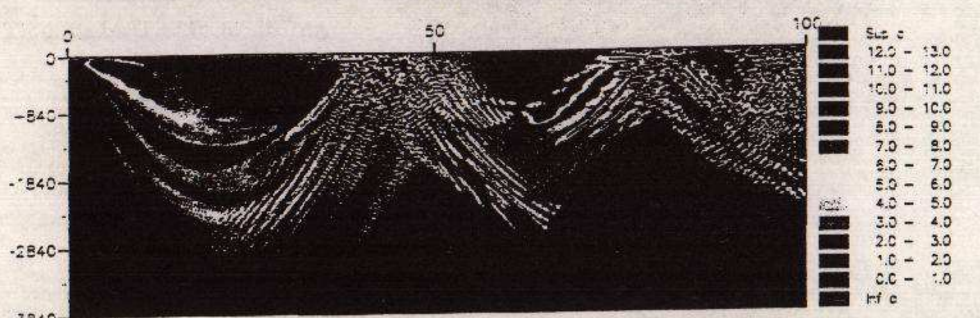
RANGE (km)



Sound intensity in a deterministic medium



Mean sound intensity in a random medium



Difference of intensity levels of the two first pictures

Figure (3):

Comparison of intensity levels obtained with a statistical approach (Paralea) and a stochastic resolution (Modalea)

F = 50 Hz, Source: Hs= 50m, Bottom = 200m, observed at 100m.