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THE CONVERGENCE BEHAVIOUR OF A MULTICHANNEL ACTIVE NOISE CONTROL SYSTEM.

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1. INTRODUCTION

The use of active techniques in the control of periodic sound is well documented [1,2,3 and 4]. The process involves the introduction of a number of secondary sound sources whose outputs are arranged so as to destructivly interfere with some unwanted primary sound field. A number of sensors are used to measure the performance of the system and allow it to track any changes in the primary field.

The control problem is thus how to adapt the outputs of a number of sources so as to achieve the desired response at the sensors. The multichannel LMS algorithm [1,2] has proven to be very effective in many practical situations [3,4] being both relatively quick and robust. This algorithm is iterative and uses a gradient estimate method to seek out the desired solution.

This paper presents both experimental and computer simulation results for such a control system with 32 microphones and 16 loudspeakers. It then goes on to explain (with the aid of further simulations) how the convergence behaviour of the system arises. The trade off between control effort and reduction and the effects of errors are also discussed.

2. THEORY

For convenience we choose the variables usually employed in acoustic active control, with the secondary sources fed by complex elements of the vector $\mathbf{q}$ and the error sensors giving an output represented by the complex elements of $\mathbf{p}$. The matrix $\mathbf{Z}$ models the acoustic response of the enclosure and, in a practical system, the response of the anti-aliasing and reconstruction filters. Finally $\mathbf{p_p}$ represents the error sensor response due to the primary field alone. These quantities are related by

$$\mathbf{p}(\omega_0) = \mathbf{p}_p(\omega_0) + \mathbf{Z}(\omega_0)\mathbf{q}(\omega_0) \tag{2.1}$$

We now choose a cost function or error criterion which we wish to minimise, in this case since we wish to reduce the sound pressure level we choose to minimise the sum of the squared outputs of the error sensors, which may be written as

$$J = \mathbf{p}^H\mathbf{p} = \mathbf{p}_p^H\mathbf{p}_p + \mathbf{p}_p^H\mathbf{Z}\mathbf{q} + \mathbf{q}^H\mathbf{Z}^H\mathbf{p}_p + \mathbf{q}^H\mathbf{Z}^H\mathbf{Z}\mathbf{q} \tag{2.2}$$

in which the superscript H denotes the Hermitian transpose. The explicit dependence on ω_0 has been dropped for convenience. This is a standard quadratic function of $\mathbf{q}$ and for the simple case of one secondary source we can visualise an 'error surface' in the shape of a bowl. Because $\mathbf{Z}^H\mathbf{Z}$ is a positive definite matrix the error surface will have a unique global minimum.

Assuming the number of error sensors is greater than the number of secondary sources then the problem is overdetermined, and has a solution which minimises J given by

$$\mathbf{q}_{opt} = -[\mathbf{Z}^H\mathbf{Z}]^{-1}\mathbf{Z}^H\mathbf{p}_p \tag{2.3}$$

Combining equations 2.1, 2.2 and 2.3 gives minimum error criterion

$$J_{min} = \mathbf{p}_p^H[\mathbf{I} - \mathbf{Z}[\mathbf{Z}^H\mathbf{Z}]^{-1}\mathbf{Z}^H]\mathbf{p}_p \tag{2.4}$$

Since the primary field may be changing and the calculation of the inverse of $\mathbf{Z}^H\mathbf{Z}$ is computationally expensive (and possibly ill conditioned) we now seek an iterative approach to minimising J.

Since the cost function J is a quadratic function of each of the available variables (the real and imaginary parts of the components of $\mathbf{q}$), gradient descent methods, if properly implemented and stable, should converge to the global minimum of J i.e., the optimum least squared solution, J_{min}.

The frequency domain multichannel steepest descent algorithm can be written in the form [7]

$$\mathbf{q}(k) = \mathbf{q}(k-1) - \alpha\mathbf{Z}^H\mathbf{p}(k-1) \tag{2.5}$$

where α is the convergence coefficient. The convergence of the vector of source strengths will now be analysed, following the approach of Widrow and Stearns [2].

If we now expand $\mathbf{Z}^H\mathbf{Z}$ into $\mathbf{Q}\Lambda\mathbf{Q}^H$ where $\mathbf{Q}$ is the unitary matrix whose columns are eigenvectors of $\mathbf{Z}^H\mathbf{Z}$ and Λ is the diagonal matrix of eigenvalues of $\mathbf{Z}^H\mathbf{Z}$ and then define a rotated and translated set of coordinates of the error surface given by

$$\mathbf{v} = \mathbf{Q}^H(\mathbf{q} - \mathbf{q}_{opt}) \tag{2.6}$$

Combining equations 2.3 and 2.5 gives

$$[\mathbf{q}(k) - \mathbf{q}_{opt}] = [\mathbf{I} - \alpha\mathbf{Z}^H\mathbf{Z}][\mathbf{q}(k-1) - \mathbf{q}_{opt}] \tag{2.7}$$

Assuming we start with no output from the secondary sources i.e., $\mathbf{q}(0) = 0$, then

$$[\mathbf{q}(k) - \mathbf{q}_{opt}] = [\mathbf{I} - \alpha\mathbf{Z}^H\mathbf{Z}]^k[-\mathbf{q}_{opt}] \tag{2.8}$$

and so we can now write the update equation, in terms of the principal coordinates of the error surface as

$$\mathbf{v}(k) = [\mathbf{I} - \alpha\Lambda]^k\mathbf{v}(0) \tag{2.9}$$

in which the matrix in brackets is diagonal, so that each component of $\mathbf{v}(k)$ converges independently in accordance with

$$v_m(k) = (1 - \alpha\lambda_m)^k v_m(0) \tag{2.10}$$

where v_m is the mth diagonal element in Λ, i.e., the mth eigenvalue of $\mathbf{Z^H Z}$. Note also that all $v_m(k)$ will tend to zero (i.e., $\mathbf{q}(k) \rightarrow \mathbf{q}_{opt}$) with increasing k, provided that

$$|1 - \alpha\lambda_m| < 1 \quad \text{i.e.,} \quad 0 < \alpha < \frac{2}{\lambda_m} \qquad \forall m \tag{2.11}$$

The maximum value for α is thus limited by the largest eigenvalue, λ_{max} such that

$$0 < \alpha < \frac{2}{\lambda_{max}} \tag{2.12}$$

Also, for small $\alpha\lambda_m$ we have that $(1-\alpha\lambda_m)^k \approx e^{\alpha\lambda_m k}$ and the time constant of convergence associated with the mth eigenvalue is $\tau_m \approx \frac{1}{\alpha\lambda_m}$, the slowest 'mode' of convergence is thus associated with the smallest eigenvalue λ_{min}: $\tau_{max} \approx \frac{1}{\alpha\lambda_{min}}$, and since $\alpha \lesssim \frac{1}{\lambda_{max}}$ then

$$\tau_{max} \gtrsim \frac{\lambda_{max}}{\lambda_{min}} \tag{2.13}$$

and τ_{max} is large when there is a large "eigenvalue spread".

3. EXPERIMENTS

The active control system built for the experiments on active control of propeller-induced cabin noise [3] consisted of a system using 32 microphones and 16 loudspeakers. The system uses digital filters working on the in-phase and quadrature parts of the reference signal to drive the loudspeakers. These filters coefficients thus compare directly with the real and imaginary parts of the complex quantities described above. The sampling frequency of the system (f_s) was 704Hz. See reference [6] for more details.

The microphones and loudspeakers were set up in a large wooden enclosure measuring $2.2m \times 2.2m \times 6m$, the walls of which were lined with foam to increase the acoustic damping. The microphones were set out in a regular 4 by 8 grid at standing head hight (1.7 metres) and the loudspeakers were evenly distributed at both floor and ceiling hight. These positions were chosen for convenience and not to take advantage of any particular acoustic properties of the enclosure. The primary field was supplied by another, larger loudspeaker set away from the others at a height of 0.5 meters.

The primary loudspeaker was excited with a pure tone of 88 Hz producing a primary level ($\mathbf{p_p}$) of 107.8 dB. The convergence coefficient of the system was adjusted, through trial and error, to be as high as possible. When the control system was turned on the level was reduced to 73.9 dB after approximatly 10 minutes (a reduction of 33.9 dB). The convergence

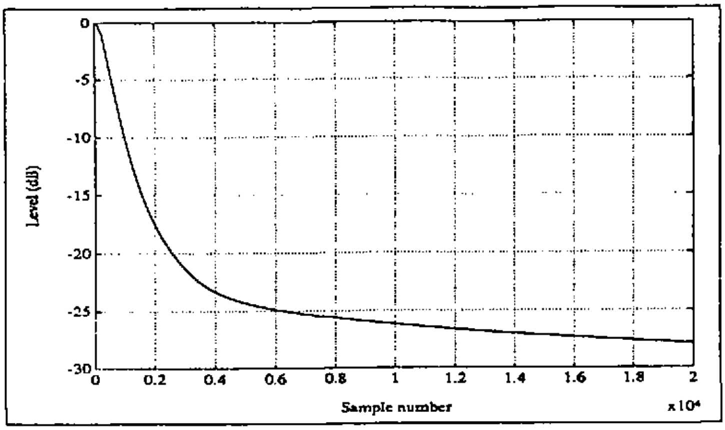

Figure 3.1 The convergence of a 32 microphone, 16 loudspeaker active control system relative to the primary (uncontrolled) level at 88 Hz.

of this experiment, over some 28 seconds, is shown in figure 3.1 and can be seen to exhibit a classic 'double slope' behaviour [2] whereby the bulk of the control is performed very quickly in the initial 'fast-mode' and then the system spends a considerable amount of time removing the last few decibels in the second 'slow-mode'. Note that in this figure and all subsequent figures the sum of squares of errors is calculated relative to the primary field and is hence a direct measure of the reduction.

This two-mode behaviour is described by Widrow and Stearns [2] in terms of the convergence of the filter coefficients along 'valleys' in the performance bowl corresponding to very small eigenvalues. It was, however, noted that the ratio of the largest to the smallest eigenvalue of $\mathbf{Z}^H\mathbf{Z}$ was much larger than the ratio of the slopes of the two "modes" observed in practice.

4. FURTHER ANALYSIS

We now seek to explain this convergence behaviour in terms of the measured properties of the system. The cost function (equation 2.2) can be expressed as

$$J(k) = J_{min} + [\mathbf{q}(k) - \mathbf{q}_{opt}]^H[\mathbf{Z}^H\mathbf{Z}][\mathbf{q}(k) - \mathbf{q}_{opt}] \quad (4.1)$$

or in terms of the principal coordinates

$$J(k) = J_{min} + \mathbf{v}^H(k)\Lambda\mathbf{v}(k) \quad (4.2)$$

Substituting $\mathbf{v}(k) = [\mathbf{I} - \alpha\Lambda]^k\mathbf{v}(0)$ into equation 4.2 and commuting the diagonal matrices, we obtain

$$J(k) = J_{min} + \mathbf{v}^H(0)[\mathbf{I} - \alpha\Lambda]^{2k}\Lambda\mathbf{v}(0) \quad (4.3)$$

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$$\therefore \quad J(k) = J_{min} + \sum_{m=1}^{M} |v_m(0)|^2 \lambda_m (1 - \alpha\lambda_m)^{2k} \tag{4.4}$$

Assuming $\mathbf{q}(0) = 0$ so that $\mathbf{v}(0) = -\mathbf{Q}^H \mathbf{q}_{opt}$ and noting that

$$\mathbf{q}_{opt} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^H\mathbf{Z}^H\mathbf{p}_p \tag{4.5}$$

where use has been made of $[\mathbf{Q}\Lambda\mathbf{Q}^H]^{-1} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^H$ then it follows that

$$\mathbf{v}(0) = -\Lambda^{-1}\mathbf{Q}^H\mathbf{Z}^H\mathbf{p}_p \tag{4.6}$$

If we now consider a singular value decomposition of the transfer impedance matrix

$$\mathbf{Z} = \mathbf{R}\Sigma\mathbf{Q}^H \tag{4.7}$$

so that $\mathbf{Z}^H\mathbf{Z} = \mathbf{Q}\Sigma^T\Sigma\mathbf{Q}^H = \mathbf{Q}\Lambda\mathbf{Q}^H$ as above. Then we may express the initial conditions in the principal coordinates as

$$\mathbf{v}(0) = -[\Sigma^T\Sigma]^{-1}\Sigma^T\mathbf{R}^H\mathbf{p}_p \tag{4.8}$$

where

$$\mathbf{R}^H\mathbf{p}_p = \underline{\varepsilon} = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_L]^T \tag{4.9}$$

which is the primary field transformed in terms of the eigenvectors of $\mathbf{Z}$ at the microphones, therefore

$$v_m(0) = -(\lambda_m)^{-\frac{1}{2}}\varepsilon_m \qquad \text{for } m = 1 \text{ to } M \tag{4.10}$$

and hence

$$J(k) = J_{min} + \sum_{m=1}^{M} |\varepsilon_m|^2 (1 - \alpha\lambda_m)^{2k} \tag{4.11}$$

Note that provided $|1 - \alpha\lambda_m| < 1 \quad \forall m$, then $J(k)$ converges with increasing k, to the value J_{min}, which can itself be expressed in terms of the singular value decomposition of $\mathbf{Z}$:

$$J_{min} = \mathbf{p}_p^H\mathbf{R}[\mathbf{I} - \Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T]\mathbf{R}^H\mathbf{p}_p \tag{4.12}$$

giving

$$J_{min} = \sum_{m=M+1}^{L} |\varepsilon_m|^2 \tag{4.13}$$

So if the primary field is transformed into components

$$[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M, \ldots, \varepsilon_L]^T = \mathbf{R}^H\mathbf{p}_p \tag{4.14}$$

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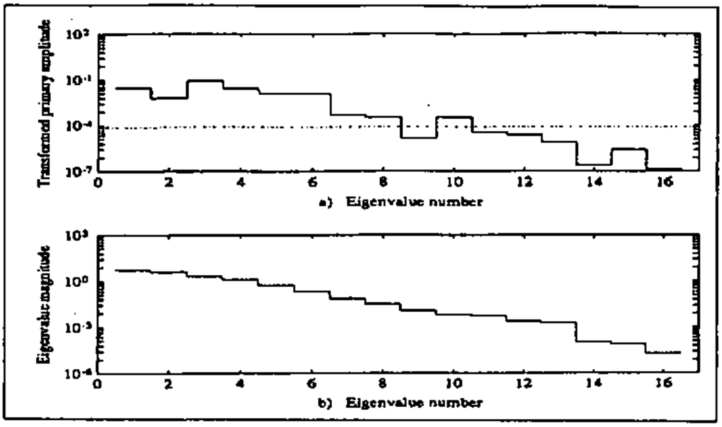

Figure 4.1 (a) The primary field transformed into the principal coordinates of the control system, $\mathbf{R}^H\mathbf{p}_p$. (b) The eigenvalues of $\mathbf{Z}^H\mathbf{Z}$.

The first M components are removed by the active control system, leaving the remaining $L - M$ components as residues.

Figure 4.1(a) shows $\mathbf{R}^H\mathbf{p}_p$, the initial levels of the modes, note that the third, first and fourth are the largest, also shown (as a dashed horizontal line) are the sum of the last sixteen modes which is equal to the residual field. Figure 4.1(b) Shows the eigenvalues of $\mathbf{Z}^H\mathbf{Z}$ which, as we have seen, determine the convergence rate of the modes.

Equation 4.11 (the convergence expressed in terms of the principal coordinates) was simulated using the data from the LMS simulation. The convergence is shown in figure 4.2. Figure 4.2 also shows how this convergence can be 'broken down' into the sum of M (sixteen in this case) curves corresponding to the individual control modes. This simulation does not contain the delays in the physical system but does give a good prediction of the convergence, which suggests that in this case convergence is limited by eigenvalue spread rather than physical delay.

Equation 2.13 gave an estimate of the fastest convergence time however, it can now be seen that this is an over-simplification since, once transformed into the principal coordinates, we see that the convergence time depends not only on the speed of convergence but also on the level of excitation of the modes of the control system by the primary source. Further the effective λ_{max} may well result from the sum of several modes whose individual contributions are significantly less than the total. Similarly the effects of the λ_{min} will be negligible if the corresponding primary level is well below the residual field and in general it would be more accurate to choose a mode (or sum of modes) who significantly effect the shape of

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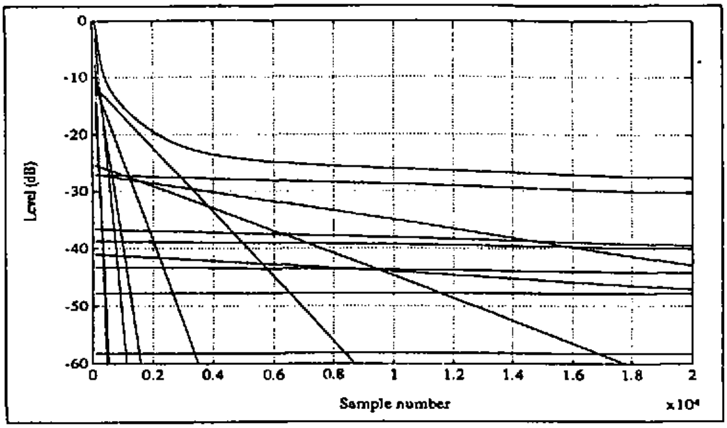

Figure 4.2 The 16 individual convergence curves corresponding to the control modes and the total sum of the modes plus the residual field.

the convergence curve.

We can define the control "effort" as the sum of the squares of the vector of secondary source strengths, $\mathbf{q}^H(k)\mathbf{q}(k)$. Plotting Reduction against effort, (figure 4.3) shows the way that the effort increases significantly as the system approaches the optimum value. This increase can become very pronounced with different configurations and primary fields and indicates that it may not always be desirable or practical to obtain the optimum reduction.

When the measured transfer function matrix $\mathbf{Z}$ is a bad model of the physical environment then the convergence of the system often resembles that of figure 4.4 where the initial convergence appears to be the same as before but then over a long period of time the level increases until it reaches some final steady state value. If the modelled transfer function matrix is denoted $\hat{\mathbf{Z}}$ then this steady state value will be given by

$$\mathbf{q}_{\text{steady state}} = -[\hat{\mathbf{Z}}^H\mathbf{Z}]^{-1}\hat{\mathbf{Z}}^H\mathbf{p} \qquad (4.15)$$

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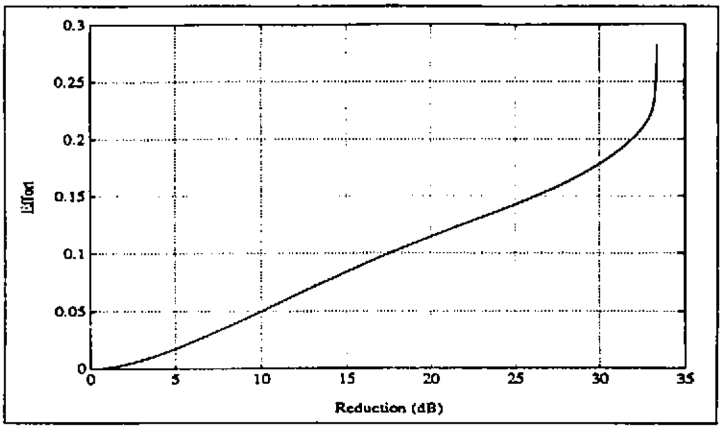

Figure 4.3 The increase in reduction against control effort.

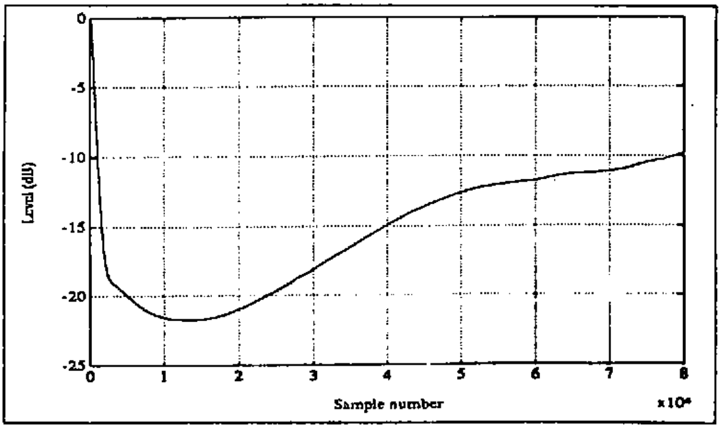

Figure 4.4 The convergence of the control system with errors in the transfer function measurements.

5. CONCLUSIONS

A multichannel active control scheme has been presented along with measured results from a 32 microphone, 16 loudspeaker system. Simulations have also been presented which show that the delays had very little effect on the systems' convergence and this behaviour has been explained in terms of the principal coordinates of the control problem. This has also shown that previous estimates on the fastest achievable convergence time do not take into account enough of the problem (specifically the primary field) to be accurate. A more complete analysis of the convergence time has also been presented.

6. BIBLIOGRAPHY

[1] S. J. Elliott, I. M. Stothers and P. A. Nelson, "A multiple error LMS algorithm and its application to the active control of sound and vibration," *IEEE transactions on acoustics, speech and signal processing*. ASSP **35**, (1987), 1423–1434.

[2] B. Widrow and S. Stearns, "Adaptive signal processing," *Prentice-Hall*, (1985).

[3] S. J. Elliott, P. A. Nelson, I. M. Stothers and C. C. Boucher, "Preliminary results of in-flight experiments on the active control of propeller-induced cabin noise," *Journal of Sound and Vibration*, **128** (1989), 355-357.

[4] S. J. Elliott and I. M. Stothers, "A multichannel adaptive algorithm for the active control of start-up transients," presented at Euromech **213**, Sept. 1986.

[5] S. Haykin, "Adaptive filter theory," *Prentice-Hall*, (1986), 105.

[6] S. J. Elliott, P. A. Nelson, I. M. Stothers and C. C. Boucher, "In-flight experiments on the active control of propeller-induced cabin noise." to be published in *Journal of Sound and Vibration*, (1990).

[7] S. J. Elliott, "Lecture notes for the ISVR short course on active noise and vibration control", (1989).