THE MUSICAL SAW

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INTRODUCTION

When an ordinary hand-saw is bent into a S-shape, an interesting acoustical effect can occur. Tapping the blade of the saw will reveal that beyond a critical degree of curvature, a very lightly-damped vibration mode appears, confined in the vicinity of the point of inflection of the S. Even though the saw blade is held in the hands, the damping of this mode is so low as to be comparable with the intrinsic internal damping of the steel of which the blade is made. This confined mode of vibration can be excited with a violin bow, and the resulting wailing sound is what is usually described as the "musical saw". Our concern here is not with this continuous excitation, however, but with the mechanism of the confinement and the nature of the confined mode or modes of vibration.

To investigate this question, we can idealise the saw. Teeth and a handle are certainly not necessary, and neither, it turns out, is a tapering blade. We thus consider a strip of uniform width and thickness (small compared with the width), bent along generators running across the width into a shape with a radius of curvature varying in some specified way along the length. We will first consider the vibration-transmission properties of such a strip with constant curvature (i.e. a section of cylindrical shell), and then use this information to see what happens when the curvature varies slowly with distance.

WAVEGUIDE MODES ON A CYLINDRICAL STRIP

Harmonic wave propagation along a section of cylindrical shell such as shown in Fig. 1 can be studied most easily using the theory of shallow-shell bending [1]. We are then presented with a waveguide problem. It is possible to satisfy the boundary conditions along the edges of the strip provided the normal displacement across the width of the strip takes one of a discrete series of shapes, closely resembling free-free beam vibration modes. Because of the mirror-symmetry of the whole system, these shapes are alternately symmetric and antisymmetric about the centre-line of the strip. (The two in-surface components of motion must also be included in the theory, and these obey slightly different symmetry conditions. However, we need not go into such details for this short account.) For each of these waveguide modes, the important information about wave propagation is contained in the dispersion relation, the variation of axial wavenumber with frequency.

A computer program has been written to calculate the first few waveguide modes and their dispersion characteristics. It is based on Rayleigh's method using power-series trial functions, in a similar way to the computations described in Ref. [2]. The two classes of modes, symmetric and antisymmetric about the centre-line, are computed separately. Results are given here only for the symmetric modes. Figure 2 shows the computed cross-sectional shapes for the first three of these modes, in each case the shapes for ten different values of axial wavenumber being superimposed. It is immediately apparent that the shapes do indeed resemble free-free beam functions, and that they are not very sensitive to wavenumber. Further computations reveal that they are also not very sensitive to curvature. This insensitivity adds credibility to the discussion of varying curvature to be given later, based on an assumption of slow variation.

The dispersion curves for these three waveguide modes are given in Figure 3 for a variety of curvatures of the strip. The solid lines correspond to zero curvature, in other words to a flat strip. The behaviour revealed is just as one would guess. The lowest mode describes bending-beam motion of the whole strip, as can be seen from the displacement function in Fig. 2(a), which is approximately uniform across the width. The corresponding dispersion curve shows the characteristic parabolic shape associated with bending beams or plates. It starts from the origin — there is no cut-on frequency for this mode. The higher waveguide modes have cut-on frequencies, corresponding to cross-section resonance with the appropriate shape, with no variation along the length of the strip. From these cut-on frequencies the curves rise monotonically, eventually asymptoting to the bending-beam curve as axial bending comes to dominate the potential energy of the motion when the wavelength becomes short compared with the width of the strip, so that the precise cross-direction shape ceases to matter much.

The curves plotted in progressively shorter dashes correspond to increasing curvature of the strip, as specified in the caption. Towards the right-hand side of the picture, curvature is seen to have little effect. On the left-hand side, though, there are very dramatic effects of curvature on the second mode, and rather less dramatic effects on the third mode. The bending-beam mode is hardly altered at all. The second mode, which will turn out to be responsible for the audible confined mode of the saw, has its cut-on frequency raised very dramatically by curvature, and with the higher values of curvature it assumes a shape which is no longer monotonic with wavenumber. The falling portion of the curve then indicates a situation where the group velocity has the opposite sign to the phase velocity.

The peculiar shape of the dispersion curve for the second mode with high curvature can be explained, qualitatively at least, quite easily. For the flat plate, the vibrations are inextensional at all frequencies and wavelengths. Introducing curvature necessitates some extensional motion in almost all cases, and thus raises frequencies to a greater of lesser extent depending on the amount of such motion needed. For the lowest, bending beam, mode, very little is needed. If the motion were exactly constant across the width of the strip, the motion would be exactly inextensional. This cannot quite be so because of edge effects associated with Poisson's ratio, but the raising of frequency is minimal. For the higher modes, though, it is easy to visualise that there must be substantial amounts of extensional motion when the wavelength is very long, and a dramatic rise in frequency is the result. As the wavelength decreases, the motion can adjust so that less longitudinal extension is needed, although at the expense of some in-surface shearing motion. This is the reason for the falling portion of the dispersion curve. With even shorter wavelengths the rising trend associated with bending takes over, and extensional effects cease to matter much.

SLOWLY VARYING CURVATURE

We can now use Fig. 3 to investigate the effect of slowly varying curvature. As stated earlier, the confinement effect appears when the saw is bent into an S-shape. This has zero curvature at the inflection point, and curvature increasing monotonically in magnitude away from that point in both directions. For the simple model of a uniform beam in static equilibrium under end moments, the deformed shape would be a cubic function of length, so that the curvature would vary linearly along the saw.

Suppose now that we drive the strip at a frequency corresponding to the horizontal line labelled A in Fig. 3, in such a way as to excite the second symmetric waveguide mode only. At the centre, where the curvature is zero, this mode can propagate with a wavelength corresponding to the point labelled 1. As it travels away from the centre, it encounters steadily increasing curvature. The frequency is fixed, so the wavelength must vary so that at each point the dispersion relation appropriate to the local curvature is satisfied. From the behaviour of the family of dispersion curves we see that this cannot continue indefinitely. The wave will reach a point with a curvature corresponding to the first dashed curve in Fig. 3, labelled 2, and by then the wavelength has become infinite. Beyond that point, the curvature is too high for a wave of this type to propagate at all, so our wave must be reflected in some way near this critical point, and travel back towards the inflection point. It then passes through, and meets increasing curvature again on the other side. It will thus reflect again from the corresponding critical point, and repeat the process. We see immediately that trapped modes are possible by this mechanism, at frequencies such that the total phase increase on a round trip is an integer multiple of 2π.

The qualitative behaviour of the saw is thus explained. There is a critical degree of curvature such that the wave reflects before it reaches either of the points at the ends of the saw by which it is being held. Below that, while there will still be modes of vibration in the usual way, they will be highly damped. Once this internal reflection process isolates a mode from the ends of the saw, its damping is governed only by internal dissipation in the material, plus radiation damping from any sound produced. The first trapped mode is the most tightly-confined, so at first we get just one such mode. With more curvature, we can confine higher modes of the same waveguide type. Eventually, we can also confine modes corresponding to higher waveguide branches. This effect is clearly audible on a real saw, by tapping and increasing the curvature. Also, as we increase the curvature we raise the frequency of the first trapped mode. This is because of the increased potential energy associated with longitudinal bending, since the length-scale of the confinement is reduced. This effect is one means whereby the pitch of the saw is varied during normal playing. (A second means, in practice, concerns the taper of the saw blade. By moving the inflection point to wider or narrower parts of the saw lower or higher pitches are produced, for a given degree of curvature.)

Simple Chladni-pattern observations confirm that the note of the saw in its usual state does indeed correspond to the second symmetric waveguide mode, with two nodal lines running along the length of the saw. With more curvature, modes can become confined with three and four lengthwise nodal lines. Notice that the lowest, bending beam, waveguide mode cannot produce confinement by this mechanism, since Fig. 3 shows that its dispersion characteristic is essentially independent of curvature. Such waves travel round corners quite happily, without any need for reflection.

A more detailed theoretical study, which we do not go into here, reveals that the "hand-waving" account given above of the confinement process holds to greater accuracy than one might suppose. The process of reflection would in general require some end correction, since one would not expect the wave to reflect with no phase change precisely from the point of critical curvature just identified. However, a proper analysis by slowly-varying theory reveals that this is precisely what does happen — the end correction turns out to be zero!

If we turn again to Fig. 3, we can repeat our discussion for the higher frequency corresponding to the horizontal line labelled B. We are now in the region of the diagram where the dispersion curves display a falling then rising shape, and this has some implications for the confined modes. This regime may not be of much significance for the real musical saw as usually played, but it offers an interesting vibration phenomenon which might repay further study. Our wave this time leaves the point of inflection with the wavelength given by the point labelled 3. It then travels unimpeded until it reaches the rather high curvature corresponding to the point labelled 4, at a minimum of the curve. Here it must reflect, as before. In general, this reflection process might be expected to produce a mixture of waves corresponding to the two possible wavelengths at the given frequency, to the left and to the right of the minimum point 4. However, theory reveals that this is not the case — all the wave energy goes into the branch of the curve to the left of the minimum. Thus as the wave travels back towards the centre of the saw, it now traverses the falling portion of the dispersion curves, until it reaches another critical point where the fixed frequency corresponds to the cut-on for this waveguide branch. Here it reflects again, this time heading back outwards. It goes out to the first critical point, and reflects back in again, now on the rising portion of the dispersion characteristic to the right of the point 4. This time it can propagate all the way back to the centre, and out the other side to repeat the whole complicated process there. The resulting trapped modes should thus exhibit a rather complicated spatial variation, and it would be most interesting to set up an experiment to try to observe it in practice. This we have not yet done.

References

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[2] M.E.McIntyre and J. Woodhouse. On measuring wood properties, Part 2. J. Catgut Acoust. Soc. 43, 18-24 (1985)

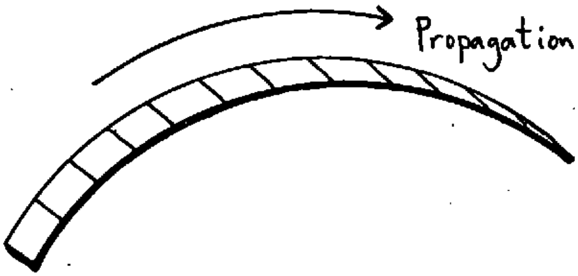

Figure 1. Sketch of a section of cylindrical shell.

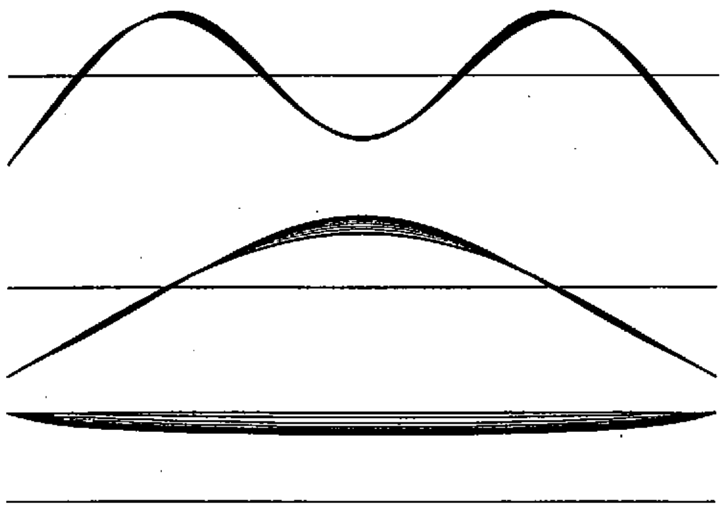

Figure 2. Cross-sectional shapes in the first three symmetrical waveguide modes. The lowest, bending-beam mode is at the bottom. The horizontal lines show the zero datum in each case. Each curve is plotted for ten different axial wavenumbers, covering the range plotted in Fig. 3.

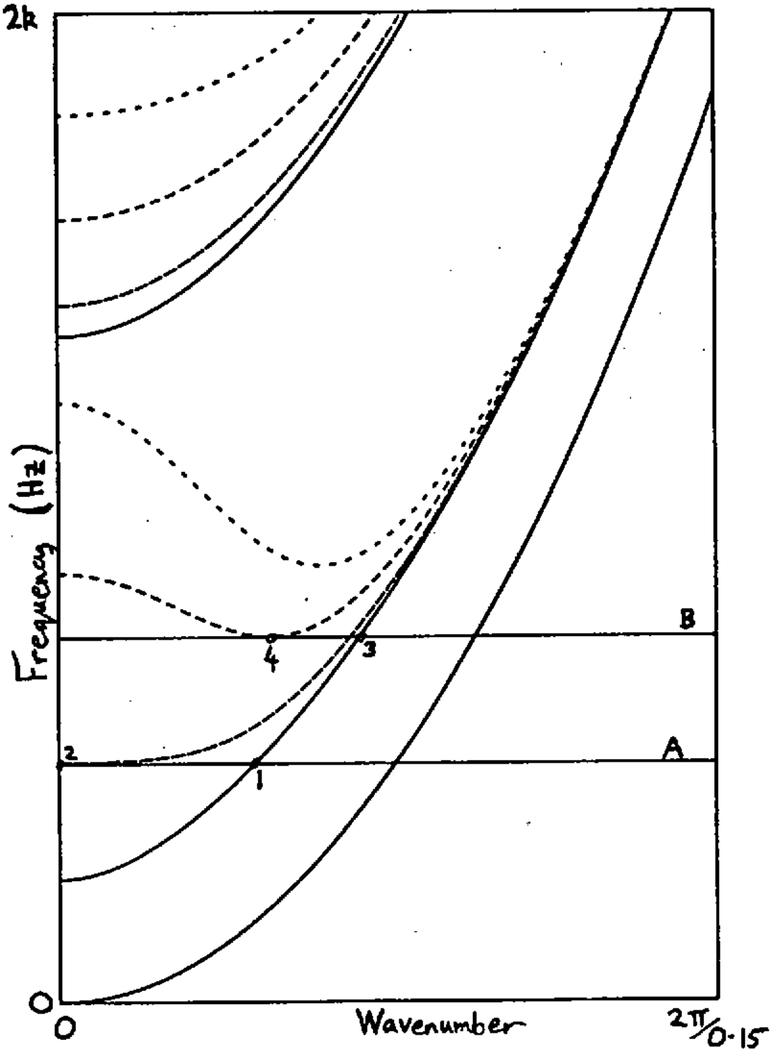

Figure 3. Dispersion curves for symmetrical waveguide modes on a section of cylindrical shell. Solid lines show a flat plate, dashed lines progressively curved plates with radii of curvature 2m, 1m and 0.7m respectively. The strip is of width 150mm, thickness 1mm, and has the Young's modulus and density of mild steel. The horizontal lines are discussed in the text.

THE CARIBBEAN STEEL DRUM

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The instruments of a Caribbean steel band, made from old oil drums and called "pans", are perhaps the most implausible, superficially, of all musical instruments. However, there is more to them than at first appears, and since they are becoming increasingly popular in schools but are still expensive as they are individually hand-made, there is good reason to investigate how they work with a view to speeding the process of manufacture. Here we consider the linear vibrational behaviour of a pan as the natural first stage in such an investigation, while recognising that a full understanding of the subtleties of tone quality in practice might require going beyond this at a later stage.

A pan is made by dishing the top surface of an oil drum ("sinking") into a shape which is roughly a portion of a sphere, then banging the individual note areas back to give regions which are approximately flat, with elliptical outline. Each region is then tuned to its specified pitch. In common with some other tuned percussion instruments, like church bells or tympani, more than one overtone frequency is tuned in each region, to give a very strong sense of pitch. For the pan we have measured, every note has a fundamental and an octave, then a mode which is always tuned to a musically-consonant interval (ranging from a whole tone to a fifth for different notes), then a fourth mode which is sometimes tuned to a double octave.

Some obvious vibrational questions, then, are as follows. What forms of shell curvature allow strong confinement of several vibration modes to a localised area of the pan? What are the forms of the mode shapes in question? How, by adjusting the details of the shape of each region, can three or four modes be tuned to specified pitches? Is the process significantly influenced by variations in thickness as well as in shape? Are there important vibrational effects due to metallurgical differences between pans, arising from composition and/or the precise history of cold working and heat treatment during the manufacturing process? We will give preliminary answers to the first two of these questions, and comment on how the others might be approached. Notice, incidentally, that this problem differs from most problems in musical acoustics in the welcome sense that we can at least <u>ask</u> some sensible questions about the physics of the vibrations at the outset. Usually any question of real interest to makers or players of instruments is barricaded behind psychoacoustical problems of the physical correlates of subjective judgements. In this case, the questions above all arise from issues of direct interest to pan-makers. The last of the questions is perhaps the most important to them. A significant proportion of oil drums prove, at a rather advanced stage in the manufacturing process, to be incapable of making good pans, and have to be thrown away. If this wastage could be eliminated by some preliminary test which could detect "bad" drums at the outset, a useful cost saving would result immediately.

The phenomenon of confinement of vibration modes to parts of a smoothly-varying shell structure occurs in other contexts as well. One example is the so-called musical saw, discussed elsewhere in this volume. In that case, some explicit theoretical analysis could be given, but in the present case only a qualitative argument is available as yet. The more complicated geometry in this case will probably necessitate numerical rather than analytical methods to give substance to this argument. So far as it goes, though, it is quite convincing. It relies on the famous account by Lord Rayleigh [] of inextensional vibration. If it is kinematically possible for a given shell structure to move in such a way that there are no in-surface strains anywhere on the middle surface of the shell, then such a motion is likely to approximate closely to one of the low-frequency vibration modes of the shell. The essential reason is that if extensional strains are present, they carry a very large penalty in terms of strain energy compared with that associated with pure bending motion. Thus an inextensional motion is likely to be very close to a local minimum of the total strain energy, and so by Rayleigh's principle will be a good approximation to a mode shape.

The tuned regions of a pan are, to a first approximation, flat areas which blend smoothly into uniformly curved surroundings on all sides. A vibration confined to such a flat region will be inextensional (for small displacements), whereas any motion which extends into the curved regions will of necessity involve extension as well. We might thus expect to find modes approximating to those of the flat regions alone, with some kind of fixed boundary conditions around the edges — perhaps simply-supported boundaries would make a good first guess.

To see whether this guess is anywhere near the truth, we have carried out an investigation of three of the areas on our pan by holographic interferometry. We have chosen the three "A" regions, tuned nominally to 220Hz, 440Hz and 880Hz. (It should be pointed out that this particular pan has gone rather out of tune since it was made, presumably as a result of various indignities inflicted on it in the name of science. It is due for a re-tune.) These regions appear in a vertical line from the centre to the bottom in the picture of the pan in Fig. 1. Figure 2 shows the first four modes of the notes whose nominal pitches are 220Hz and 440Hz, with the corresponding frequencies. Figure 3 shows a series of higher modes confined, in the main, to the region A220. Finally, the two interferograms in Fig. 4 show what happens at higher frequencies. Modes are no longer confined to individual regions, but extend throughout the pan. This is to be expected: as frequency rises, so the energy associated with bending motion increases, and the argument for inextensionality loses its force.

The mode shapes revealed by Figs. 2 and 3 are in good agreement with the inextensionality argument given above. Modes are very strongly confined to roughly elliptical areas, except where resonant coupling occurs to modes of other notes on the pan. Careful examination reveals that the different modes in a given region do not all stop on precisely the same boundary curve: the lowest frequency in each case is the most strongly confined, as the inextensionality argument suggests and as was found also in the musical saw. This phenomenon can be confirmed readily by tapping the pan near the edges of a given region. The note corresponding to each confined mode ceases quite abruptly when the tapping point leaves its area of confinement, and it is possible to tap at points where the fundamental of a given note is absent but the higher overtones are still excited.

The sequence of modes in each of the two regions shown in Fig. 2 is just as one would expect. The lowest frequency involves vibration of the whole region, the next two have a single node line running approximately along the minor and major axis of the ellipse respectively, and so on. For both A notes here, we see that the interval between first and second modes is indeed an octave (allowing for the fact, mentioned above, that this pan is rather out of tune). The interval between the second and third modes is a major third for the note A220, and a minor third for A440. Both notes then have a double-octave between fourth and first modes, and in the case of the note A220 this fourth mode is clearly seen to resonate with the octave mode of the 440Hz region and the fundamental of the 880Hz region in the centre. Such resonant coupling no doubt plays a role in the overall tone quality of a well-tuned pan.

Now we know the mode shapes, we can begin to see how the fine-tuning process might be carried out. The pitch of the fundamental and the octave of a given area can be controlled by varying the total area and the aspect ratio of the elliptical region. To get a third mode into tune with these two without upsetting either of them, the maker would seem to have rather little freedom of manoeuvre. We certainly need to go beyond the simple model of a flat, elliptical region with simply-supported boundaries to discuss the issue. This accords with the comments of a maker about the relative difficulties of tuning the three modes in a given region. Fine adjustments to the shape, deliberate departures from mirror-symmetry and flatness, and so on, are needed to tune the third mode, and fourth mode if that is tuned too.

A possible way to investigate this theoretically would be to measure in digital form the shape (and perhaps the thickness distribution) of a few patches on a real, well-tuned pan. This geometrical information could then be used to fit some kind of finite-element mesh to the shape, so that low modes could be computed and compared with experiment. If that comparison proved satisfactory, one could use the finite-element solutions thus obtained to do a perturbation calculation, to see how the first few mode frequencies are each affected by a small shape change in a given small area. One might then be able to see where to hammer to achieve a desired change in the relative frequencies of the modes. We hope to attempt such an investigation in the near future.

Another thread of the investigation which we should consider is that of the metallurgical causes of "bad" pans. Some preliminary measurements on samples of such rejected pans suggest that high levels of manganese sulphide in the steel might be partly to blame. The connection may be to do with an effect on the internal damping of the material, perhaps in association with the rather high levels of plastic deformation which occur during the process of turning an oil drum into a pan. The chemical composition of the mild steel from which oil drums are made is only controlled within quite wide limits, so that significant variations in levels of this and other impurities occur between different batches of steel. Work in this area is being actively pursued.

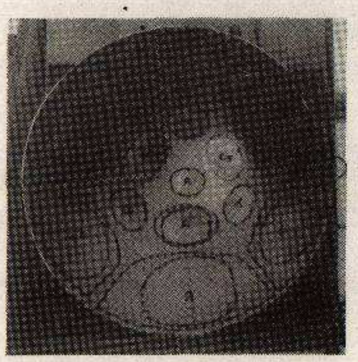
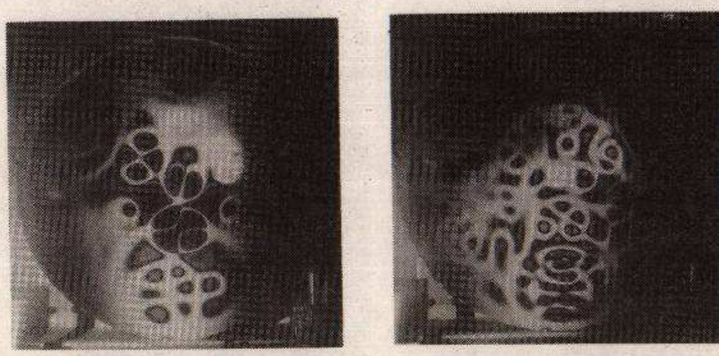


Figure 1. A general view of the pan studied here. The three 'A' regions investigated in detail run in a vertical line down from the centre. Each note area is shown by two lines, the score line visible on the surface, and within it the roughly elliptical area wherein the vibration is actually confined.

Figure 2

Figure 3



1822 Hz 2963 Hz

Figure 4. Two modes at higher frequencies, showing that confinement to the designated note areas has broken down and the vibration patterns cover large areas of the pan.