Bearing Errors Due to Correlated Noise

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This problem was attempted in order to explain an anomaly in the results from a particular sonar set under certain conditions. It failed to explain that anomaly, but may possibly be useful in estimating some performance characteristics.

Consider a simple model of a hypothetical split beam sonar as shown in Figure 1. Four elements are arranged equally spaced on a horizontal line, with spacing d say. The left hand pair are summed, and the right hand pair are summed, phase shifted by 90^{o} and time delayed. The two signals are then correlated. The phase shifting converts a peak in the correlation function, due to the presence of a target, into a zero crossing, whose position is easier to measure, as a function of time delay.

Consider half beams whose axis is perpendicular to the array (i.e. to time delay between the elements of a pair). In the presence of noise which is uncorrelated between any pair of elements the time delay at which a zero crossing of the correlation function occurs gives a direct measure of the target bearing. However, if the noise field is correlated between pairs of elements the measured bearing is biased by the noise.

The bias and variance of bearing in the presence of an isotropic plane wave gaussian noise field are calculated and compared with the effects of an uncorrelated noise field, which produces variance but no bias.

The calculations prove to be cumbersome and a particular case is considered where both signal and noise are bandlimited white noise waveforms in the frequency range B to 2B with the element spacing d equivalent to $\lambda/2$ at the upper frequency 2B.

Mathematical Model

The mathematical model of the system is as follows. Both signal and noise waveforms are assumed to be bandlimited, white, gaussian and stationary.

The signal at element j is for a target at θ_s (see Figure 2) where

$$s_j(t) = s\left(t - jd\cos\theta_s/c\right) \quad (1)$$

$$\langle s(t)\, s(t+\tau)\rangle = \sigma_s^2\, W(\tau) \quad (2)$$

$\langle\cdot\rangle$ means ensemble average and

$$W(\tau) = \frac{\sin \pi B\tau}{\pi B\tau} \cos 2\pi f_0 \tau \quad (3)$$

with B the bandwidth and f_0 the centre frequency.

Thus σ_s^2 is the received signal power at each element.

The noise at element is

$$n_j(t) = \frac{1}{(4\pi)^{1/2}} \int d\Omega \; n\left(t - \frac{jd\cos\theta}{c}, \theta, \phi\right) \qquad (4)$$

where

$$\langle n(t,\theta,\phi)\, n(t+\tau,\theta',\phi')\rangle = \sigma_n^2 W(\tau)\, \delta(\cos\theta - \cos\theta')\, \delta(\phi - \phi') \qquad (5)$$

This corresponds to isotropic plane wave noise, with no correlation between different directions. Also σ_n^2 is the received noise power at each element.

Let P be the phase shift operator. Then it may be shown that

$$\langle s(t) P s(t+\tau)\rangle = \sigma_s^2\, Q(t) \qquad (6)$$

with

$$Q(t) = \frac{\sin \pi B\tau}{\pi B\tau} \sin 2\pi f_o \tau \qquad (7)$$

A similar result holds for $n(t)$.

Define

$$S_L(t) = s_o(t) + s_1(t) \qquad (8)$$

$$N_L(t) = n_o(t) + n_1(t) \qquad (9)$$

$$S_R(t) = P\{s_2(t) + s_3(t)\} \qquad (10)$$

$$N_R(t) = P\{n_2(t) + n_3(t)\} \qquad (11)$$

Then the wavenumber output from the left channel is $S_L(t) + N_L(t)$, and that from the right channel is $S_R(t) + N_R(t)$. It is assumed that the signal and noise do not correlate.

Then the correlator calculates the function $C(u)$ with

$$C(u) = \int_0^T dt\, \big(S_L(t) + N_L(t)\big)\big(S_R(t) + N_R(t)\big) \qquad (12)$$

For a target at θ_B, with $\tau = d\cos\theta_B / c$,

$$\frac{\langle C(u)\rangle}{T} = \sigma_s^2 \{ Q(u-\tau) + 2Q(u-2\tau) + Q(u-3\tau)\}$$

$$+ \frac{\sigma_n^2}{4\pi} \int d\Omega \left\{ Q\left(u - \frac{d\cos\theta}{c}\right) + 2Q\left(u - \frac{2d\cos\theta}{c}\right) + Q\left(u - \frac{3d\cos\theta}{c}\right)\right\} \qquad (13)$$

The noise term integral may be evaluated in terms of the tabulated function $Ci(X)$, and so $\langle C(u)\rangle$ may be plotted as a function of u, given B, fo, d, θ_B, σ_s^2 and σ_n^2.

For uncorrelated noise, the noise term drops out of $\langle C(u) \rangle$, and because $Q(x)$ is an odd function of x, the zero crossing of the signal term is at $u = 2\tau = \frac{2d\cos\theta_s}{c}$ thus giving the target position. Of course, this is the reason for doing the phase shift on one half beam.

However, in the presence of correlated noise the noise term does not vanish. It does have a zero at $u = 0$ because $Q(x)$ is odd but in general will not be zero at $u = 2\tau$. The result is that the zero crossing of $\langle C(u) \rangle$ will lie between 0 and 2τ and so given a biased target bearing. This zero crossing may be plotted as a function of τ.

For illustration a particular case has been chosen which does show significant bias effects and is, in some sense, typical of some sonars. For bandwidth B, the centre frequency will be taken to be $3B/2$, so that the upper frequency limit is 2B, and the lower one, B.

The interelement spacing is $\lambda/2$ at the upper frequency 2B, so that at the lower frequency the spacing is only a quarter wavelength.

Useful simple approximations may be made to both signal and noise terms. The signal term may be expanded in powers of $(u-2\tau)$ and τ, and the noise term in powers of u. It turns out that the lowest order approximations are adequate for $0 < u < 2\tau$ provided τ is such that the target lies within 10° of the normal to the array, which is equivalent to $B\tau < 1/20$ for the stated interelement spacing.

The result is

$$\langle C(u) \rangle = BT\left\{12\pi\sigma_s^2(u-2\tau) + \frac{8\sigma_n^2}{9\pi}u\right\}. \qquad (14)$$

Thus $\langle C(u) \rangle$ has a zero crossing at

$$u - 2\tau = -2\tau\left(1 + \frac{27\pi^2\sigma_s^2}{2\sigma_n^2}\right)^{-1} \qquad (15)$$

So that a (S/N) ratio of -21dB would bias the result to the mid point of a bearing normal to the array and the actual bearing, irrespective of the BT product. Figure 3 shows a plot of apparent bearing against actual bearing.

Now consider the <u>variance</u>.

This will give a measure of the uncertainty of the position of the zero crossing of C(u).

The variance V(u) is given by

$$\langle C^2(u) \rangle - \langle C(u) \rangle^2$$

V(u) is thus a double integral of a product of 4 gaussian variables. There is a well-known result on the expectation value of such products, such that for zero mean variables

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle\langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle\langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle\langle x_2 x_3 \rangle \quad (16)$$

Using this

$$V(u) = \int_0^T dt \int_0^T dt' \Big\{ \langle S_L(t) S_L(t') \rangle\langle S_R(t+u) S_R(t'+u) \rangle$$

$$+ \langle S_L(t) S_R(t'+u) \rangle\langle S_L(t') S_R(t+u) \rangle + \langle N_L(t) N_L(t') \rangle\langle N_R(t+u) N_R(t'+u) \rangle$$

$$+ \langle N_L(t) N_R(t'+u) \rangle\langle N_L(t') N_R(t+u) \rangle + \langle S_L(t) S_L(t') \rangle\langle N_R(t+u) N_R(t'+u) \rangle$$

$$+ \langle S_L(t') S_R(t+u) \rangle\langle N_L(t) N_R(t'+u) \rangle + \langle S_L(t) S_R(t'+u) \rangle\langle N_L(t') N_R(t+u) \rangle$$

$$+ \langle S_R(t+u) S_R(t'+u) \rangle\langle N_L(t) N_L(t') \rangle \quad (17)$$

The expectation values can be written down from previous equations and the integrand depends only on $t-t'$, not on t and t' separately. By a change of variables, and writing $\xi = t'-t$, the variance is easily shown to be of the form

$$V(u) = \int_{-T}^{T} (T - |\xi|)\, F(\xi)\, d\xi \quad (18)$$

where $F(\xi)$ is small for $\xi \gg B^{-1}$ so that if $BT \gg 1$ the term in $|\xi|$ is very small compared to that involving T and may be neglected. Now there is no harm in extending the limit of integration to infinity to give

$$V(u) = T \int_{-\infty}^{\infty} F(\xi)\, d\xi \quad (19)$$

The expection values of second order products of S and N all involve the functions W and Q, and V(u) may be simplified further using the results

$$\int_{-\infty}^{\infty} W(z-x) W(x+y)\, dx = \frac{1}{2B} W(z+y) \quad (20)$$

and

$$\int_{-\infty}^{\infty} Q(z-x) Q(x+y)\, dx = \frac{-1}{2B} W(z+y) \quad (21)$$

Thus V(u) may be evaluated in terms of Si and Ci functions and is complicated. It is of the form

$$V(u) = C_{SS}\sigma_S^4 + C_{SN}\sigma_S^2\sigma_N^2 + C_{NN}\sigma_N^4 \quad (22)$$

where the C's depend on u, and τ.

To simplify the result, consider a target on the beam axis, $\tau = 0$, and calculate the variance only at u = 0, i.e. at the zero crossing of Then after some tedious calculations, under the conditions of a signal band of white noise from B to 2B with element spacing $\lambda/2$ at 2B,

$$V(0) \simeq \frac{T}{B}\left(10.1\,\sigma_s^2\sigma_n^2 + 3.3\,\sigma_n^4\right) \tag{23}$$

If it is assumed that the slope of C(u) in the neighbourhood of the zero crossing is unchanged from its expectation value (or at any rate that any change is a higher order effect) the variance on the zero crossing itself may be deduced. Instead of $\tau = 0$ a value for 2τ will be found whose standard deviation is given by

$$2\tau_s \simeq \frac{\left(V(0)\right)^{1/2}}{\frac{d}{du}\langle C(u)\rangle\Big|_{u=0}} \tag{24}$$

$$\simeq \frac{1}{B(BT)^{1/2}} \cdot \frac{\left(10.1\,\sigma_s^2\sigma_n^2 + 3.3\,\sigma_n^4\right)^{1/2}}{\left(12\pi\,\sigma_s^2 + (8/4\pi)\sigma_n^2\right)} \tag{25}$$

$$< 6.42\; B^{-1}\,(BT)^{-1/2} \tag{26}$$

For uncorrelated noise, with u = 0, $\tau = 0$

$$V(0) = TB^{-1}\left\{2\sigma_n^4 + 8\sigma_s^2\sigma_n^2\right\} \tag{27}$$

It is not necessary to locate the particular frequency band or specify the element spacing to obtain this result. Thus the variance in the case of uncorrelated noise is a little less than that found for correlated noise having the particular band location and element spacing used here.

The approximations for very small (S/N) ratios break down because of the assumption that the slope of each actual C(u) curve is identical to that of $\langle C(u)\rangle$ close to its zero crossing.

To clear up this point the covariance rather than just the variance needs to be examined.

Figure 4 shows plots of the standard deviation as a function of correlated signal to noise and also the standard deviation for uncorrelated noise for BT = 10^5.

Conclusions

A mathematical model has been proposed to find the bias in the bearing indicated by a split beam correlator in the presence of isotropic plane wave bandlimited noise. The standard deviation of bearing has also been calculated and compared with that due to uncorrelated noise.

FIGURE 1

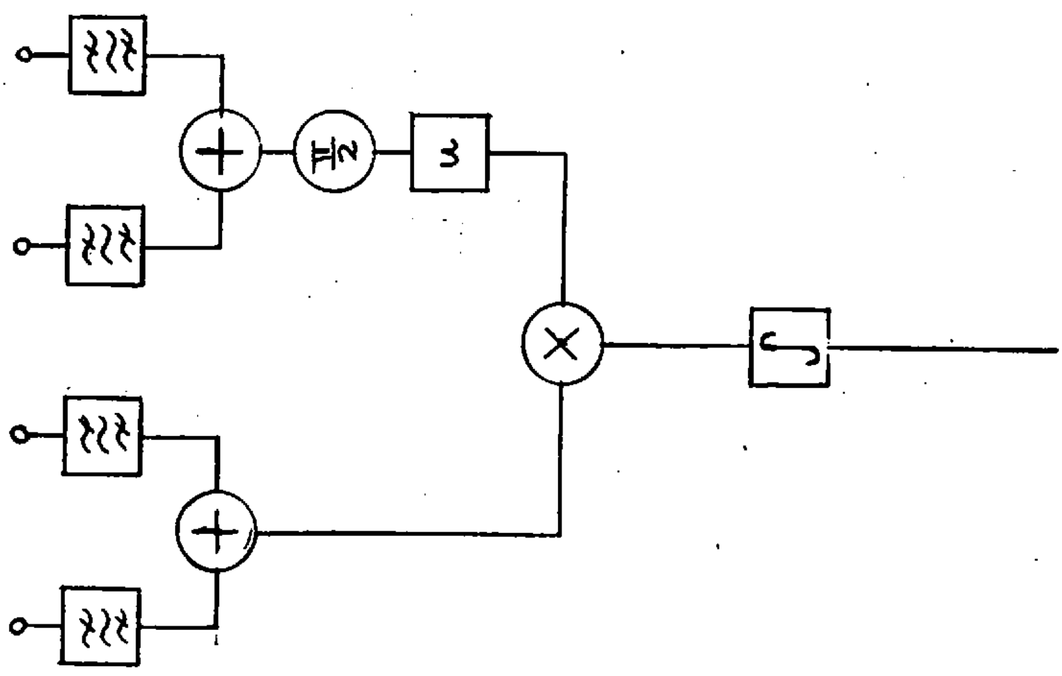
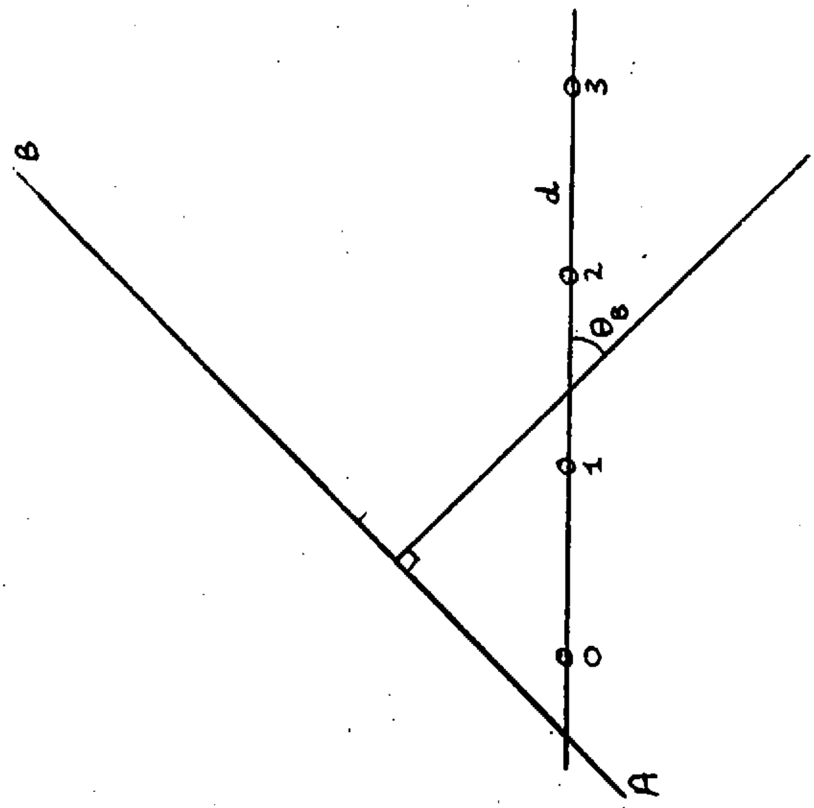


FIGURE 2



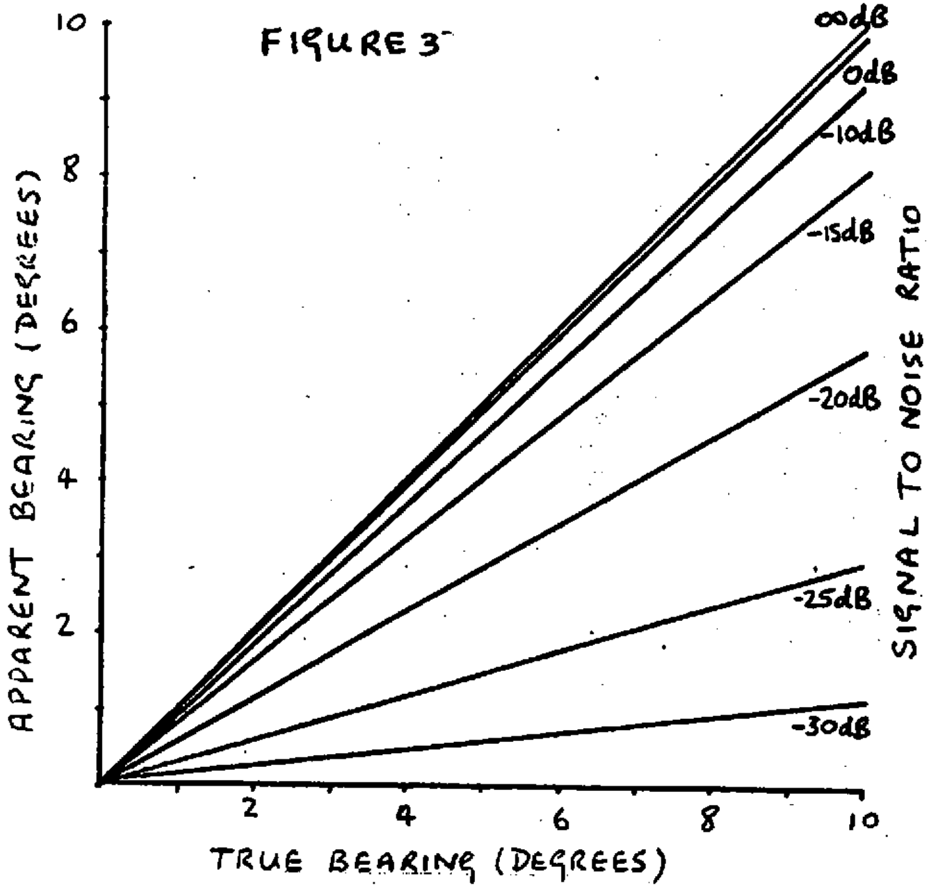


FIGURE 3
APPARENT BEARING (DEGREES)
TRUE BEARING (DEGREES)
SIGNAL TO NOISE RATIO
∞dB
0dB
-10dB
-15dB
-20dB
-25dB
-30dB
10
8
6
4
2
2
4
6
8
10

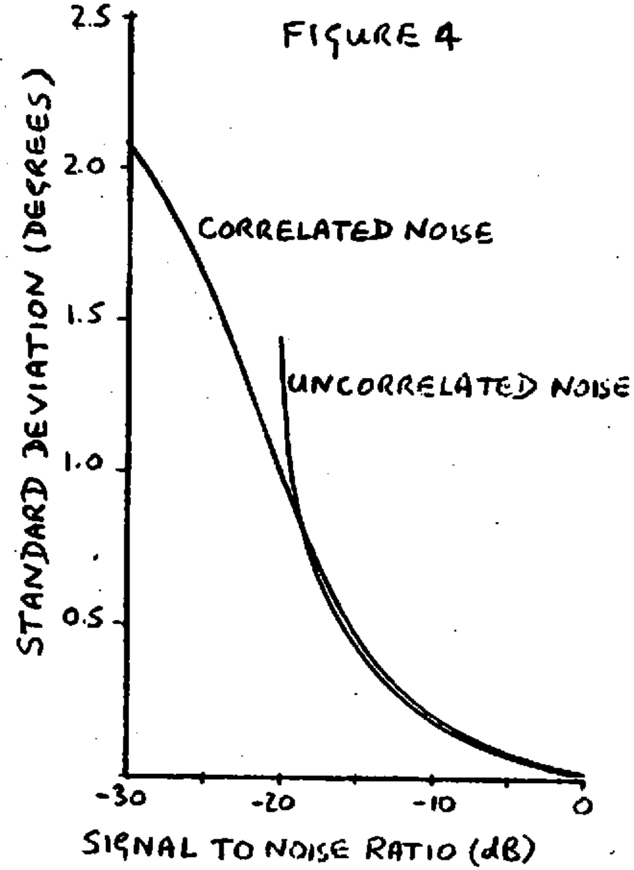


FIGURE 4
STANDARD DEVIATION (DEGREES)
SIGNAL TO NOISE RATIO (dB)
CORRELATED NOISE
UNCORRELATED NOISE
2.5
2.0
1.5
1.0
0.5
-30
-20
-10
0