WAVEFRONT CURVATURE OF NEAR FIELD SOURCES BY VECTOR ROTATIONS.

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1 INTRODUCTION

This paper discusses the possibility of direction-finding using the signal-subspace or spatial cross-spectral density matrix (SCSDM) without performing a pre-conceived search through the array manifold. It is advantageous in using all the information contained in the original data matrix or from an eigen-decomposition performed on the SCSDM. With present technology arrays may be several kilometres in length and therefore it is no longer inconceivable that sources lie purely in the far field. This means that near and far field sources have to be treated equally and the curvature of near field wavefronts revealed, allowing further processing to be performed that may locate the sources.

The approach followed in [1] finds the wavefronts by rotating eigenvectors taken in pairs from the whole signal space until a suitably defined cost function converges. In a similar manner the method described in [2] rotates eigenvectors individually. The intent of this present paper is to show how all the signal subspace eigenvectors can be rotated simultaneously.

The problem in [1] and [2] is related to that of mixed near and far field emitters being received by an ideal linear array, where the spatial phase distributions are very much direction and range dependent. In particular, the signal received from an emitter in the near field experiences a non-linear time delay progression between adjacent transducers. Time domain methods which estimate the inter-element propagation delays from peaks in the cross-correlation function are commonly employed in this situation. However there are few papers that focus on the signal processing required to extract this information from the frequency domain.

The effect of spreading will be taken as negligible, which puts a lower limit on how close an emitter may be to the array, but this is not considered to be a serious shortcoming. It is the observation that an acoustic source is received with equal power on each sensor that forms the basis for the algorithms presented here.

2 SIGNAL AND NOISE MODELS

The observed spatial snapshot $\mathbf{x}(k)$ is assumed to have come from a simple model for M uncorrelated sources in isotropic white noise. The signals are received by a linear equispaced array having P transducers.

$$\mathbf{x}(k) = \mathbf{D}\mathbf{s}(k) + \mathbf{n}(k) \tag{1}$$

The columns of matrix D contain the M wavefront phase delay vectors while $\mathbf{s}(k)$ and $\mathbf{n}(k)$ are the vectors of source and noise complex amplitudes respectively.

The spatial cross-spectral density matrix (SCSDM) is given by

$$\mathbf{R} = \mathbf{D}\mathbf{C}\mathbf{D}^H + \sigma_N^2 I \tag{2}$$

where $\mathbf{C} = E[\mathbf{s}(k)\mathbf{s}^H(k)]$ is the diagonal source cross-power matrix and $E[\,]$ is the statistical expectation operator. Since the cross-power matrix is diagonal it may conveniently be combined with the matrix D so that $\bar{\mathbf{A}} = \mathbf{D}\sqrt{\mathbf{C}}$ This allows the covariance matrix to be written as

$$\mathbf{R} = \bar{\mathbf{A}}\bar{\mathbf{A}}^H + \sigma_N^2 I \tag{3}$$

It is seen that the columns of $\bar{\mathbf{A}}$ are simply the source rms amplitudes multiplied by their respective array phase responses.

Performing an eigen-decomposition on the SCSDM results in

$$\mathbf{R} = \mathbf{V}\mathbf{Z}\mathbf{V}^H \tag{4}$$

where $V = [v_1, \ldots, v_P]$ are the eigenvectors and $Z = \mathrm{diag}[\zeta_1, \ldots, \zeta_P]$ are the eigenvalues arranged in descending order of magnitude.

If an infinite number of snapshots are taken, it can be shown that the smallest $P - M$ eigenvalues are all equal to the noise power σ_N^2. This allows one to rewrite this identity in an equivalent form to Eq.(3)

$$R = V(Z - \sigma_N^2 I)V^H + \sigma_N^2 I$$

$$= BB^H + \sigma_N^2 I \tag{5}$$

where B contains only those eigenvectors associated with the largest M eigenvalues weighted by the square root of their noise corrected eigenvalues.

In practice the smallest eigenvalues will not be equal so a maximum likelihood estimate of the isotropic noise power can be calculated from their mean.

Comparing equations 3 and 5 one can see that $\bar{A}$ and B are related by an $M \times M$ unitary transformation T in the following manner.

$$\bar{A} = BT \tag{6}$$

Notice that the columns of B are orthogonal whilst those of $\bar{A}$ are not. Also each column in $\bar{A}$ has the form of a constant multiplied by a vector of complex exponentials, so the magnitude of each element in a column is identical.

Our objective is to find this transformation matrix, using purely magnitude information contained in the structure of matrix $\bar{A}$. By ignoring phase when determining T then the issue of whether an emitter is in the near or far field is irrelevant. However, having found the transformation, one can then examine A and determine the phase.

3 OBTAINING WAVEFRONT VECTORS DIRECTLY FROM THE SIGNAL SUBSPACE

The approach followed in [2] attempts to find each column in T sequentially by minimizing this cost function

$$J = |\Gamma - |BT||^2 \tag{7}$$
$$\Gamma = 1_P f^T$$

where 1_P is a Px1 vector of *ones* and f contains the estimated source rms amplitudes. These can be found from averaging down each column of $|BT|$. In this text $|*|$ is the element-wise magnitude operator and $|*|$ is the Frobenius norm.

By minimizing this cost function the eigenvalue weighted eigenvectors are rotated to become rms amplitude weighted wavefront vectors. In [3] it is shown that *minimizing* Eq.(7) is functionally equivalent to *maximizing* the cost function given below

$$J = \sum_{m<n=1}^{M} \left[\left(\frac{1}{P} \sum_{p=1}^{P} |a_{pm}^* a_{pn}|^2 \right) - \left(\frac{1}{P} \sum_{p=1}^{P} a_{pm}^* a_{pn} \right) \left(\frac{1}{P} \sum_{p=1}^{P} a_{pm} a_{pn}^* \right) \right] \tag{8}$$

where the notation $m < n = 1$ can be read as $m = 1, n = 1, m \neq n$ This rather daunting expression involves the *covariance between different columns* whereas the old cost function used the variance of individual columns.

The advantage of this cost function is that all the eigenvectors can be rotated *simultaneously* using a simple iterative algorithm whereas for the case above they had to be rotated individually. This produces far superior convergence characteristics.

This cost function can be maximized with respect to all the elements in the transformation matrix T simultaneously. As the transformation matrix is constrained to be unitary then its columns must satisfy

$$\left.\begin{array}{l} \mathbf{t}_i^H \mathbf{t}_i = 1 \\ \mathbf{t}_i^H \mathbf{t}_j = 0 \quad i \neq j \end{array}\right\} i, j = 1, \ldots, M$$

These may be appended to the cost function with un-determined Lagrangian multipliers λ_{ij}, so denote the augmented cost function by script font $\mathcal{J}$

$$\mathcal{J} = J + \left(M - \sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{ij} \mathbf{t}_i^H \mathbf{t}_j \right) \tag{9}$$

Due to the unitary constraints $\lambda_{ii}\ 1 \leq i \leq M$ must be real but $\lambda_{ij}\ \ i \neq j$ may in general be complex. Since the cost function is by definition real and positive then $\lambda_{ij} = \lambda_{ji}^*$.

To find the maximum requires differentiating this function with respect to the transformation matrix T and setting the result equal to zero. The Lagrangian constraints help confine the search path so that the optimum transformation matrix is unitary. For clarity this lengthy derivation is performed in [3] and the final result is stated here. The simplified derivative of $\mathcal{J}$ with respect to $\mathbf{t}_i$ is found to given by the i^{th} column of this matrix expression

$$\frac{\partial \mathcal{J}}{\partial \mathbf{T}} = \frac{2}{P} \mathbf{B}^H \left(\mathbf{E} - \frac{1}{P} \mathbf{F} \right) - 2\mathbf{T}\Lambda \tag{10}$$

where

$$\mathbf{E} = \mathbf{A} \odot [|\mathbf{A}|^2 (\mathbf{1}\mathbf{1}^T - I)]$$

$$\mathbf{F} = \mathbf{A}(\mathbf{A}^H \mathbf{A} - \mathbf{D}) \qquad \mathbf{D} = \mathrm{diag}[|\mathbf{a}_1|^2, \ldots, |\mathbf{a}_M|^2]$$

also $\odot$ is the element-by-element multiplication operator and Λ is the matrix of Lagrangian multipliers.

Setting this equal to zero and dividing out the factor of 2 then one arrives at the maximization condition

$$\frac{1}{P} \mathbf{B}^H \left(\mathbf{E} - \frac{1}{P} \mathbf{F} \right) = \mathbf{T}\Lambda \quad \text{or} \quad \mathbf{H} = \mathbf{T}\Lambda \tag{11}$$

Pre-multiplying both sides by $\mathbf{T}^H$ and noting that T is unitary

$$\mathbf{T}^H \mathbf{H} = \frac{1}{P} \mathbf{A}^H \left(\mathbf{E} - \frac{1}{P} \mathbf{F} \right) = \Lambda \tag{12}$$

But the i^{th} diagonal element of the left hand side is

$$J_i = \sum_{\substack{m=1 \\ m \neq i}}^{M} \left[\frac{1}{P} \sum_{p=1}^{P} |a_{pm}^* a_{pi}|^2 - \left| \frac{1}{P} \sum_{p=1}^{P} a_{pm}^* a_{pi} \right|^2 \right] \tag{13}$$

which is immediately recognized as the contribution to the cost function of the i^{th} column from A. Therefore maximizing J is the same as maximizing the trace of Λ and this is also the value of the cost function. The maximum is found by an iterative procedure which starts with an initial approximation $\mathbf{T}_1$ which gives starting values $\mathbf{A}_1 = \mathbf{B}\mathbf{T}_1$ for an approximation to the optimum value of A. Then matrices $\mathbf{E}_1$, $\mathbf{F}_1$ and $\mathbf{H}_1$ are calculated. To complete each iteration, a Hermitian symmetric positive definite matrix Λ has to be found that lies in the span of H. On pre-multiplying both sides of Eq.(11) by its transpose

$$\mathbf{H}_1^H \mathbf{H}_1 = \Lambda_1^H \Lambda_1 \tag{14}$$

then since $\mathbf{H}_1^H \mathbf{H}_1$ is positive definite symmetric it may be expressed in terms of the singular value decomposition of H

$$H_1 = UVW^H$$

$$H_1^H H_1 = WV^2W^H \tag{15}$$

where U and W are the matrices of left and right singular vectors, and V is the matrix of singular values. A Hermitian symmetric matrix in the span of H is therefore given by

$$\Lambda_1 = WVW^H \tag{16}$$

Lastly, the new iteration begins by finding a unitary transformation that satisfies the equation

$$T_2\Lambda_1 = H_1 \tag{17}$$

So substituting for the inverse Lagrange multiplier matrix

$$T_2 = H_1\Lambda_1^{-1} = UVW^H WV^{-1}W^H$$

$$= UW^H \tag{18}$$

In practice the Lagrange multiplier matrix does not have to be calculated since the updated transformation matrix is obtained directly from the SVD on H

A good initialization is required in poor signal-to-noise ratio conditions. If a direction finding algorithm such as a spatial Fourier transform or MUSIC is available then the columns of A may be estimated. This can be used to initialize T as shown below

$$T = (B^HB)^{-1}B^HA \tag{19}$$

The iterations may look expensive from a computational point of view, but it should be borne in mind that the dimensions of H are $M \times M$ where M is the number of sources, which may be small. Also the orthogonal polar decomposition UW^H of H can be calculated directly by a number fast methods, see [4], which are less computationally expensive than a full SVD. In practice very few iterations are needed and if a good initial guess at the transformation matrix is available only two or three iterations may be required. The algorithm is summarized in Appendix A.

4 PROOF OF MAXIMIZATION

The derivation above does not prove that the cost function will necessarily increase at each iteration, but this is now proved. However it does not necessarily follow that the algorithm will iterate to the global maximum, and indeed this is sometimes the case with a poor initialization. Since it is sufficient to prove that the trace of Λ increases at each iteration, then from Eq.(11) $\Lambda = T^HH$ the trace of this can be written as

$$\text{trace}(T^HH) = \text{trace}(T^HUVW^H) = \text{trace}(ZV) \tag{20}$$

where $Z = W^HT^HU$ is a unitary matrix. The trace of ZV can never be greater than the trace of V since Z is unitary, the equality comes for Z equal to the identity matrix.

$$\sum_{m=1}^{M} z_{mm}v_{mm} \le \sum_{m=1}^{M} v_{mm} \tag{21}$$

The upper bound of which is achieved when $T = UW^H$, which was the result given in the previous section. Therefore the trace of Λ is maximized by this choice of transformation.

5 OBTAINING WAVEFRONT VECTORS DIRECTLY FROM THE COVARIANCE MATRIX

This method is applicable when the sources are known to be uncorrelated but the spatial noise statistics are either unknown or cannot be estimated with any degree of confidence. It operates directly on the covariance matrix, so one is not required to perform an eigen-decomposition of this, as was necessary in section 3. Once again the covariance cost function Eq.(8) is used, since it simplifies the computational cycle and notation.

The derivation proceeds by eliminating the necessity to compute updates using the formula $A_{k+1} = BT_{k+1}$ since the matrix of eigenvectors B is unknown. Instead the expressions are manipulated so that BB^H can be replaced by the full rank covariance matrix (recall from Eq.(5) that $R = BB^H + \sigma_N^2 I$). The initialization of A is chosen in a sensible although ad-hoc fashion. It must have dimensions $P \times M$, $M < P$ so the number of sources has to be determined in advance.

Denoting the term in brackets in Eq.(11) by G then it is expressed more conveniently as

$$B^H G = T\Lambda \quad \text{where} \quad G = 1/P(E - F/P) \tag{22}$$

Pre-multiplying both sides of Eq.(22) by their transpose and noting that T is by definition unitary results in

$$G^H BB^H G = \Lambda^H \Lambda \tag{23}$$

However BB^H is the rank M signal space approximation to the covariance matrix. In this case it is now replaced by the original full rank covariance matrix.

$$G^H RG = \Lambda^H \Lambda \tag{24}$$

This matrix has dimension $M \times M$, where M is the number of sources. It is known that Λ, the matrix of Lagrangian multipliers, must be positive definite symmetric, so one is required to find a matrix of this form in the range space of $G^H RG$. The eigenvalues of $G^H RG$ will be positive semidefinite and its eigen-decomposition can be expressed as

$$G^H RG = WV^2 W^H \tag{25}$$

where W and V^2 are the matrices of eigenvectors and eigenvalues respectively. A positive definite symmetric matrix in the range space of $G^H RG$ is therefore given by

$$\Lambda = WVW^H \tag{26}$$

From Eq.(22) and Eq.(26) the transformation matrix is

$$T = B^H GWV^{-1} W^H \tag{27}$$

But also the relationship between A and B can be exploited to avoid this direct calculation of T to give the final result as follows

$$\begin{aligned} A = BT &= BB^H GWV^{-1} W^H \\ &= RGWV^{-1} W^H \end{aligned} \tag{28}$$

To determine the stopping point then, once again, the cost function is equal to the trace of Λ and this should be monitored as an indicator of convergence.

The computation involved in this method requires the eigen-decomposition of a Hermitian symmetric $M \times M$ matrix at each iteration. An algorithm summary is given in Appendix A.

6 INTERPRETING THIS MAXIMIZATION

The effect of applying this algorithm can be readily explained in that the trace of $G^H RG$ is equal to the sum of its eigenvalues. This is maximized when G is a projection matrix into the signal subspace of the covariance

matrix, in fact the optimum choice of matrix A best satisfies two conditions. The first is that each column of A lies in the signal space and the second is that the elements in each column have equal modulus. With the help of Eq.(28) this rank reduction can be explicitly shown. The covariance matrix with M sources in an arbitrary noise field can be expressed as

$$\mathbf{R} = \bar{\mathbf{A}}\bar{\mathbf{A}}^{H} + \sigma_N \mathbf{R}_N \tag{29}$$

And $\bar{\mathbf{A}}$ is the exact wavefront matrix. Therefore $\mathbf{R} - \bar{\mathbf{A}}\bar{\mathbf{A}}^{H}$ is simply the noise covariance matrix. Combining Eq.(25) and Eq.(28) one can write for the approximation A of $\bar{\mathbf{A}}$

$$\mathbf{A} = \mathbf{RG}(\mathbf{G}^{H}\mathbf{RG})^{-1/2} \tag{30}$$

hence

$$\mathbf{R} - \mathbf{AA}^{H} = \mathbf{R} - \mathbf{RG}(\mathbf{G}^{H}\mathbf{RG})^{-1}\mathbf{G}^{H}\mathbf{R} \tag{31}$$

This can be recognized as the general form of a rank reduced matrix. Therefore although one begins with an arbitrary initialization for the M columns of matrix A, Eq.(30) ensures the rank of R is reduced by M. Furthermore the cost function maximization ensures that the columns of A converge to having constant magnitude columns. This is not achieving exactly the same result as Eq.(29) which leaves a residual full rank noise term, but if the signal-to-noise ratio is high the estimation of A will be satisfactory.

7 SIMULATION RESULTS

A simulation was performed using a 32 element linear array with a 1m spacing between sensors. The array operates at 0.4 wavelength spacing to ensure the phase-unwrapping algorithm does not fail for endfire sources. The Fresnel distance for this array works out to be 380m. Four uncorrelated sources are present in the near field as shown in Figure 1, with the closest only 2.5 array lengths away. A second weaker source is almost directly under this and is masked in a conventional spatial FFT by the closer source. Two more close targets are located at the edge of the near field.

Two hundred snapshots were taken and the eigenvector rotation algorithm used to find the wavefront shapes. The transformation matrix was initialized to the identity matrix, which is a reasonable approximation if the sources are widely spaced. The signal space eigenvectors are plotted in the complex plane and it can be seen that they are indeed a good approximation to the actual wavefronts. The iterations terminated when J changed by less than 0.001%. Unwrapping the phase of matrix A resulted in the target wavefronts graph, while the hydrophone SNR graph was obtained as

$$\text{Hydrophone SNRs} = 20\log_{10}|\mathbf{A}| - 10\log_{10}\sigma_N^2 + 10\log_{10}N$$

and includes the snapshot processing gain (23dB). The noise power was taken as the mean of the 28 smallest eigenvalues.

Figure 2 shows the same situation but using the covariance matrix method. Instead of sensibly initializing the matrix A its elements were chosen from a complex gaussian random number generator. It can be seen that even with a completely random starting point the algorithm has nearly converged after 20 iterations but takes 78 iterations to stabilize to 0.001%.

There appears to be little difference in the algorithms performance until two sources are very closely spaced when the covariance matrix method outperforms the eigenvector method. This may be due to the wide eigenvalue separation for these two eigenvectors, leading to one eigenvalue dominating the cost function and slowing convergence.

8 CONCLUSIONS

Two simple iterative algoritms have been presented. One rotates eigenvectors whilst the other intrinsicly rotates vectors in the covariance matrix signal subspace. The advantage of such an algorithm is that the

sources can be in the near or far field and this will not effect the algorithm performance. The individual wavefront curvatures are revealed as are the SNR's of each source at each hydrophone. Both methods show near equal performance.

9 REFERENCES

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APPENDIX A

Eigenvector Method	Covariance Matrix Method
M = number of sources. P = number of sensors. Form initial transformation $\mathbf{T}_1 = (\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H\mathbf{A}$ $\mathbf{A}_1 = \mathbf{B}\mathbf{T}_1$ $k = 1$ repeat $\mathbf{D}_k = \text{diag}[\lvert\mathbf{a}_1\rvert^2, \ldots, \lvert\mathbf{a}_M\rvert^2]$ $\mathbf{E}_k = \mathbf{A}_k \odot [\lvert\mathbf{A}_k\rvert^2(\mathbf{1}\mathbf{1}^T - I)]$ $\mathbf{F}_k = \mathbf{A}_k(\mathbf{A}_k^H\mathbf{A}_k - \mathbf{D}_k)$ $\mathbf{H}_k = \frac{1}{P}\mathbf{B}^H\left(\mathbf{E}_k - \frac{1}{P}\mathbf{F}_k\right)$ $\mathbf{U}\mathbf{V}\mathbf{W}^H = \mathbf{H}_k$ (singular value decomposition) $\mathbf{T}_{k+1} = \mathbf{U}\mathbf{W}^H$ $\mathbf{A}_{k+1} = \mathbf{B}\mathbf{T}_{k+1}$ $J = \text{trace}[\mathbf{W}\mathbf{V}\mathbf{W}^H]$ or $\text{trace}[\mathbf{T}_{k+1}^H\mathbf{H}_k]$ $k = k + 1$ until J is stable	M = number of sources. P = number of sensors. Choose $\mathbf{A}_1 = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M]$ $(M < P)$ $k = 1$ repeat $\mathbf{D}_k = \text{diag}[\lvert\mathbf{a}_1\rvert^2, \ldots, \lvert\mathbf{a}_M\rvert^2]$ $\mathbf{E}_k = \mathbf{A}_k \odot [\lvert\mathbf{A}_k\rvert^2(\mathbf{1}\mathbf{1}^T - I)]$ $\mathbf{F}_k = \mathbf{A}_k(\mathbf{A}_k^H\mathbf{A}_k - \mathbf{D}_k)$ $\mathbf{G}_k = \frac{1}{P}\left(\mathbf{E}_k - \frac{1}{P}\mathbf{F}_k\right)$ $\mathbf{H}_k = \mathbf{R}\mathbf{G}_k$ $\mathbf{L}_k = \mathbf{G}_k^H\mathbf{H}_k$ $\mathbf{W}\mathbf{V}^2\mathbf{W}^H = \mathbf{L}_k$ (eigen-decomposition) $\mathbf{A}_{k+1} = \mathbf{H}_k\mathbf{W}\mathbf{V}^{-1}\mathbf{W}^H$ $J = \text{trace}[\mathbf{W}\mathbf{V}\mathbf{W}^H]$ $k = k + 1$ until J is stable

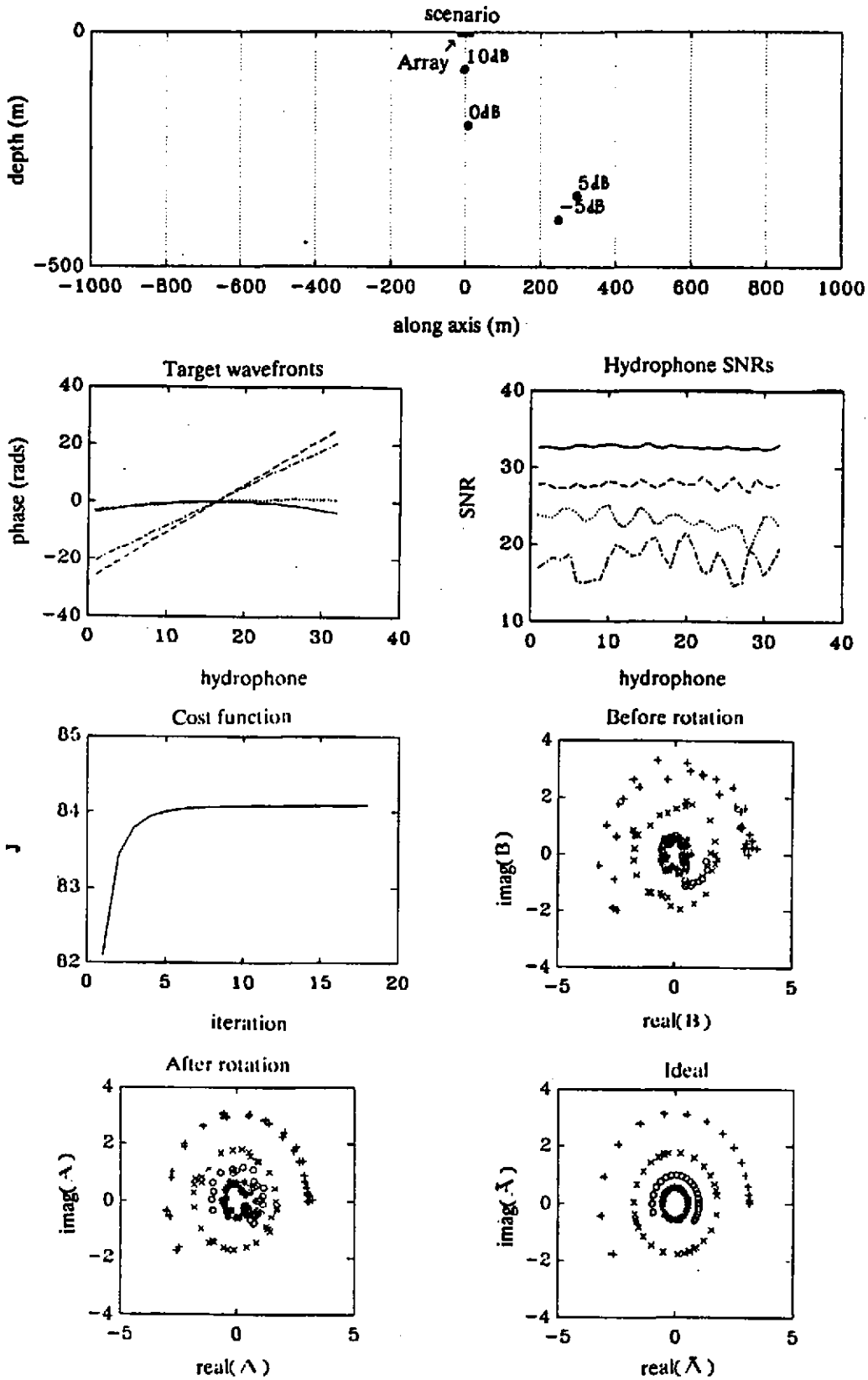

Figure 1. Algorithm performance for rotating the eigenvectors.

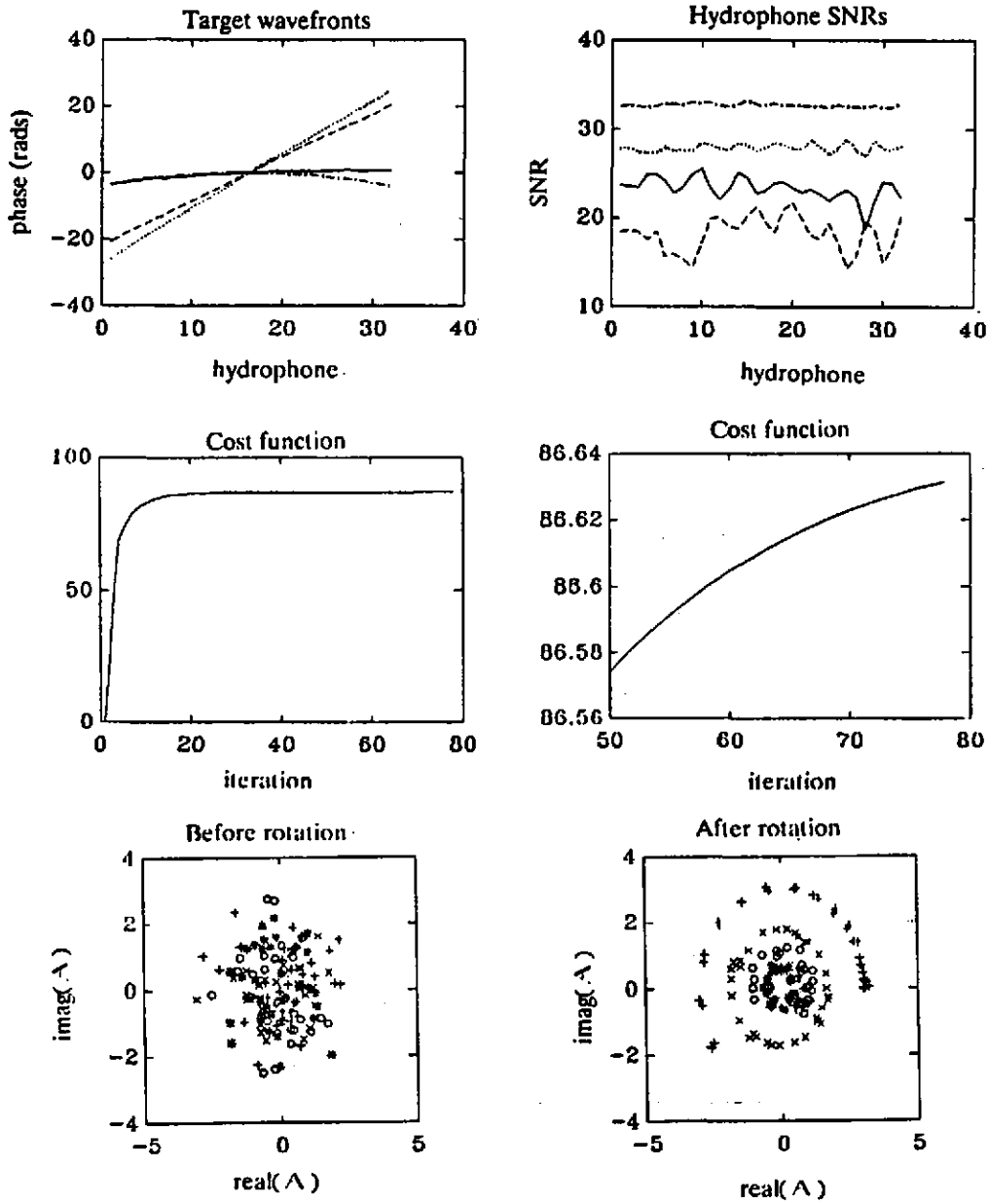

Figure 2. Algorithm performance for intrinsic rotations using the covariance matrix.