Proceedings of The Institute of Acoustics

# ON THE PROBLEM OF MAXIMUM ARRAY DIRECTIVITY

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ABSTRACT

A set of n acoustic sensors can have an ideal directivity of at least $n^2$ when arranged as a linear equi-spaced array shooting end-fire, with appropriate weightings and delays. The discussion covers the early studies by Uzkov, and the recent ones by the author on the Jacobi polynomial directionality patterns. The practicability of Jacobi arrays is illustrated, and the status of the subject is summarised. There are two major pieces of unfinished business outstanding: a proof that the directivity of n sensors cannot exceed $n^2$; and, despite a good general understanding, the linked question of a simple and fundamental principle which predicts a maximum directivity law of exactly $n^2$. Unfortunately there is no directly equivalent effect in the parallel field of spectral estimation.

## INTRODUCTION

The problem in the title is no problem until we introduce constraints. For example there is no theoretical limit on directivity if we allow a continuous spatial variation in the array sensitivity, and this is true even for an array of infinitesimal size.

Our constraint is to suppose we have n point sensors or point sources, but arranged as we wish and with any weightings and any delays. Our investigation leads us to consider end-fire line arrays, with equal spacings and equal delay increments but unequal weights. The best theoretical spacing is found to be infinitesimally small, although the paper does not set out to be one on superdirectivity.

The paper starts at the start of our subject, as introduced with aplomb by Uzkov some forty years ago [1]. We then skip over the intervening years, in which there has been only a modest amount of discussion on realisation and other questions [eg 2-5]. We take up the story by looking briefly at a current paper by the writer [6], with other relevant work by the writer and his colleagues also recently available [7-9].

The concentration here is on results and their meaning rather than on detail: calling attention to the $n^2$ or 20 log n law for directivity and asking why we should get such a good law and such a simple law.

## UZKOV

Uzkov eventually deals with a linear array of n point transducers having a spacing which is both regular and very small. He considers the response as a function of $x = \cos\theta$, where the angle $\theta$ is measured from the main lobe which has been arranged to occur at end-fire. A spherical harmonic analysis can be carried out on this response, ie it can be decomposed into the sum of n components, each corresponding to one of the first n Legendre polynomials $P_r(x)$. It is an important property of the Legendre polynomials that they are mutually orthogonal.

He now introduces a powerful theorem which is applicable to any orthogonal directional functions. If there are n sets of weights and delays giving n different and orthogonal directional patterns, with general directivity $D_r$, how should one combine these sets in order that the directivity D of the sum will be a maximum? It is easy to show (by considering variation of one set at a time) that each set should be weighted so that its main lobe response is proportional to its directivity $D_r$. (Remember to distinguish between weighting of sets and weighting of individual transducers.) The calculation of the overall D is easy because of the lack of cross-product terms in the angle integration. Thus dividing the square of the main lobe response by the sum of the angular integrations gives

$$D = \frac{\left(\sum_{r=1}^{n} D_r\right)^2}{\sum_{r=1}^{n} D_r} = \sum_{r=1}^{n} D_r \quad . \tag{1}$$

The maximum directivity of the summed sets is seen to be simply the sum of the individual directivities.

These ideas can be applied to the end-fire array and the Legendre polynomials, in order to find the upper limit to the directivity attainable. All we need to know is that for polynomial $P_r(x)$ the directivity is [1, 7]

$$D_r = 2\,r + 1 \quad . \tag{2}$$

The maximum for the polynomial sum becomes

$$D = \sum_{r=o}^{n-1} (2r + 1) = n^2 \ . \tag{3}$$

Uzkov's paper is very compact, and the part relevant to the $n^2$ law only occupies about one page. He does not bother with mundane things such as the actual pattern shape necessary to give the law, but from our rules about weighting of the sets we see that it is

$$\sum_{r=o}^{n-1} (2r + 1)\, P_r(x) = P_{n-1}^{(1,0)}(x) \tag{4}$$

where on the right-hand side we have identified the result as a Jacobi polynomial. Figure 1 illustrates the pattern for 10 sensors. It seems reasonable to call the $n^2$ array a Jacobi array, or perhaps a Jacobi-Uzkov array. Discussion on the necessary weighting of the transducers is also omitted by Uzkov, but included in the next section.

## RECENT WORK

Uzkov's attack is with a rapier, but there is another approach exemplified by the writer in Ref 6 in which the attack is with a blunt instrument. (When originally carried out it was a blissful ignorance of Uzkov's work!) But this attack is equally successful in that it succeeds in hammering the problem into submission, and the two styles of attack could be claimed to be of mutual help.

In fact Ref 6 used several methods. The initial exploration shows that for small n (n = 2, 3) the right geometry is indeed the very short end-fire line with regular spacing, and so this has also been assumed for n large. One technique uses covariance calculations, but an even more fruitful method uses angular integrations since these are simplified in the limit of infinitesimal spacing. Note that one can get (n-1) zeros or angular nulls in the pattern from n transducers. Laborious variational methods are used to experiment with the positions of these zeros in order to find the values which maximise the directivity. However it is found possible to avoid matrix calculations: solutions come virtually by inspection and are later proved correct and proved always to give $n^2$ for the directivity. These solutions turn out to be the Jacobi polynomials, as is of course inevitable.

Each zero can be associated with a dipole having a given relative delay. The angular pattern for the whole array can be obtained by a multiplication of those for the separate dipoles, and the necessary transducer weightings and delays built up by a convolution of those for the separate dipoles. This does provide a natural and practical method of calculating the transducer weightings and delays. It is found that one automatically gets equal delay increments, and weightings that are symmetrical. As the spacing approaches zero so the weight values more closely match the binomial co-efficients, though with alternating polarities. It will be appreciated that in these conditions a very short array will be very inefficient.

## STATUS

### Theory

Refs 1 and 6 both prove that an array directivity of at least $n^2$ can be achieved with n transducers for any value of n; or strictly that it can be achieved in the limit as the array size shrinks to zero. For n = 2 or 3 the answer is precisely $n^2$ for the maximum. For a regularly-spaced linear end-fire array the answer is precisely $n^2$. But for $n > 3$ it has not been proved that, by departing from the regularly-spaced end-fire array, one cannot exceed $n^2$. Achieving such a proof remains as a challenge. However there are reasons for suspecting that the regularly-spaced linear end-fire geometry may be optimum and that $n^2$ is the overall limit.

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Practice

Despite the superdirective nature of the basic design there does appear to be a useful place for arrays of the Jacobi type, or at least for those with a modest number of sensors. Consider for example a 5-element array with a quarter-wavelength spacing, which avoids the worst problems of efficiency and matching accuracy. Nevertheless the calculated directivity is only 1dB down on the ideal $n^2$ value of 25 (14 dB). In comparison with conventional arrays there is a great saving in numbers of sensors and in overall length.

## UNDERSTANDING

We now attempt to get a feel for the $n^2$ law both qualitatively and quantitatively. Leaving aside the actual proofs in Refs 1 and 6 there are many arguments which help understanding [6]. But there is room only to list them here and not to give a full discussion.

a. There is no reason from an information theoretic viewpoint why D should not greatly exceed n, remembering that the solid angle within the main lobe may be quite small.

b. A dependence of D on n (corresponding to 10 log n) is commonly met, but there are many examples of other laws.

c. The effective cross-sectional area of a receiver, projector, scatterer or absorber may greatly exceed its physical projected area.

d. The process here may be regarded as a continuation of the Hansen-Woodyard oversteering and narrowing of the end-fire beam combined with a shading of the side-lobes.

e. A superdirective array in effect makes use of the coherence of the noise field, rather than finding it disadvantageous.

f. An argument about division into angular blocks of signal, and the dispersion within such blocks, suggests the maximum D should be of order $n^2$.

g. It can be shown that the zeros should be fairly evenly spaced in $\theta$, and therefore in x-space the extreme zero will be of the order of $n^{-2}$ away from the main lobe position. Again this suggests D should be about $n^2$.

We can therefore understand quite well why D should be large and of order $n^2$. But we still hanker after finding some simple and fundamental principle which will predict exactly $n^2$!

## SPECTRAL ESTIMATION

Generally there is of course a useful parallel between array processing in the spatial domain and spectral estimation in the time domain; and we do not wish to deny a present analogy between the formation of a beam with narrow angular response and the construction of a filter with narrow frequency pass-band. But unfortunately we cannot take the analogy far since we are dealing with an end-fire beam where it must be recognised we are working in three spatial dimensions. In particular note that the wave-number resolved along the array is restricted to lie between values of ± free-space wave-number, and that the Jacobi end-fire design concentrates on one of these limits.

## CONCLUSION

The Jacobi array is important and practical, but we draw attention here to the lack of proof that the directivity limit cannot exceed $n^2$, and to the lack of a simple underlying principle explaining the precision of $n^2$.

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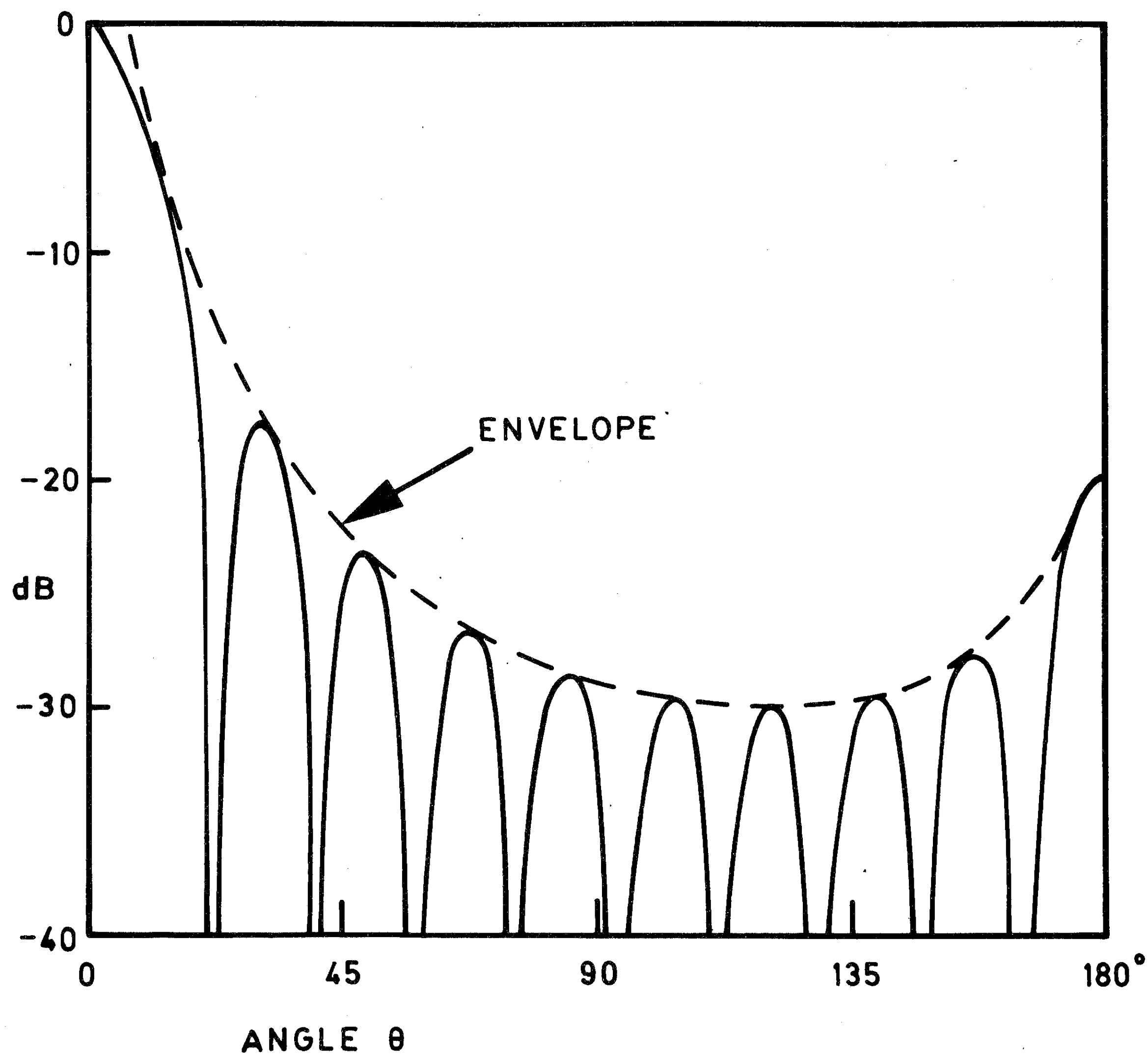


FIGURE 1 ANGULAR PATTERN FOR $P_9^{(1,0)}(\cos\theta)$ WITH 10 SENSORS