24th INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
23–27 July 2017, London

ICSV24
23-27JULY 2017
LONDON CALLING

A GENERAL METHOD FOR CALCULATING THE POWER DISSIPATED BY NONLINEAR OSCILLATORS DRIVEN BY RANDOM EXCITATION

David H. Hawes and Robin S. Langley
University of Cambridge, Deparment of Engineering, Cambridge, UK
email: dhh29@cam.ac.uk

Random excitation of mechanical systems occurs in a wide variety of structures and, in some applications, calculation of the power dissipated by such a system will be of interest. In this paper, by extending the Wiener series approach, a general methodology is developed for calculating the power dissipated by a general nonlinear multi-degree-of freedom oscillatory system excited by random Gaussian base motion of any spectrum. The Wiener series method was originally developed to compute the output of a nonlinear system to a white noise input, but is extended here to encompass a general non-white input. From the extended series a simple expression for the power dissipated can be derived in terms of the first term, or kernel, of the series and the spectrum of the input. Calculation of the first kernel can be performed either via numerical simulations or from experimental data and a useful property of the kernel, namely that the integral over its frequency domain representation is proportional to the oscillating mass, is derived. The resulting equations offer a simple conceptual analysis of the power flow in nonlinear randomly excited systems and hence assist the design of any system where power dissipation is a consideration. The results are validated both numerically and experimentally using a base excited cantilever beam with a nonlinear restoring force produced by magnets.

Keywords: Random Vibration, Nonlinear Vibration, Wiener Series, Power Dissipation

1. Introduction

In a number of applications the power dissipated by a vibrating system will be of interest. In some instances, such as vibration energy harvesting [1], the aim will be to maximise power, whereas in others, such as those involving concerns over fatigue or heat generation, the aim will be to minimise it. In addition, for single-degree-of-freedom (SDOF) systems with linear damping the power dissipated is proportional to the mean square velocity, a useful measure of the response of an oscillator.

Of particular interest here are general methods for calculating power dissipation of nonlinear systems from random excitation. In general, the most prevalent technique is to solve the Fokker-Planck equation which governs the probability density function of the response [2, 3, 4, 5]. Whilst robust, this must be achieved computationally or via simplifying the resulting equations by making assumptions. As the number of degrees of freedom increase, these solutions become significantly more involved. A noteworthy result in the case of power dissipated under white noise excitation is derived partially in [6, 2, 7, 5, 8], and more generally in [9] and shows that for a general multi-degree-of-freedom (MDOF) nonlinear system subject to white noise base excitation, the power dissipated is simply proportional to the total oscillating mass and the magnitude of the noise spectrum regardless of the specific details of the system. This result is extended in [10] which uses the Wiener series to

show that for systems exhibiting detailed balance the power dissipated under white noise excitation will be greater than or equal to the power dissipated under non-white excitation where the peak of the spectrum is taken as the magnitude of the white excitation.

The Wiener series is a useful method for analysing an output from a nonlinear system with a Gaussian white noise input via an orthogonal series expansion of the random output [11]. It is a commonly used tool for nonlinear system identification, particularly for physiological systems [12]. Whilst generally associated with white noise inputs, it can also be extended to non-white inputs [13] and this form, herein called the *extended* Wiener series, is applied in this paper. A thorough description and explanation of Wiener theory and its applications can be found in [11].

The aim of this paper is to provide a general methodology for calculating the power dissipated by a general nonlinear oscillator under non-white excitation. In what follows an introduction to the extended Wiener series is presented in Section 2 followed by the derivation of the method for calculating power dissipation in Section 3. The theory is then validated numerically and experimentally in Sections 4 and 5 respectively before conclusions are made in Section 6.

2. Wiener Series for Non-white Excitation

In this section, the extended Wiener series for non-white input excitation is introduced. The series is very similar to the Wiener series for white noise and as such, the notation of [11] is used. A nonlinear system with a Gaussian random input, $x(t)$, will produce a random output signal, $y(t)$, that can be described as a sum of functionals

$$y(t) = \sum_{n=0}^{\infty} \mathbf{g}_n[k_n; x(t)]. \tag{1}$$

When compared to the Wiener series of [11], the G-functionals, $\mathbf{G}_n[k_n; x(t)]$, have been replaced with a lower case $\mathbf{g}_n[k_n; x(t)]$ to represent that these are extended Wiener functionals for non-white noise. Each g-functional is defined as a sum of Volterra functionals, $\mathbf{K}_{j(n)}[x(t)]$, up to the order of the g-functional such that

$$\mathbf{g}_n[k_n; x(t)] = \sum_{j=0}^{n} \mathbf{K}_{j(n)}[x(t)] \tag{2}$$

where the Volterra functional $\mathbf{K}_{j(n)}[x(t)]$ takes the form

$$\mathbf{K}_{j(n)}[x(t)] = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} k_{j(n)}(\tau_1, ..., \tau_j) x(t-\tau_1)...x(t-\tau_j) \mathrm{d}\tau_1 ... \mathrm{d}\tau_j. \tag{3}$$

The order of the Volterra functional is j and $k_{j(n)}(\tau_1, ..., \tau_j)$ is called an extended Wiener kernel of order j and must be calculated. The (n) term in the subscript of both the functional and the kernel denotes that they both belong to the Volterra series of the nth order g-functional. When $j = n$ the nth order kernel $k_{n(n)}$ will be rewritten as k_n and is called the leading order kernel.

The relationship between the extended Wiener kernels in a g-functional can be found by enforcing the orthogonality condition

$$\mathrm{E}\left[\mathbf{H}_p[x(t)]\mathbf{g}_n[k_n; x(t)]\right] = 0 \qquad \text{for} \quad p < n \tag{4}$$

where $\mathrm{E}[X]$ represents taking the ensemble average of the random variable X and $\mathbf{H}_p[x(t)]$ is any Volterra functional of order p. This condition is required to create an applicable orthogonal series, like the Wiener series, that converges and where the contributions from each g-functional can be isolated in order to calculate them.

When the orthogonality condition is applied, the form of the nth order g-functional can be shown to be

$$\mathbf{g}_n[k_n; x(t)] = \sum_{m=0}^{\text{floor}[n/2]} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{(-1)^m n!}{(n-2m)!m!2^m} \int_{-\infty}^{\infty} k_n(\tau_1, ..., \tau_{2m}, \sigma_1, ..., \sigma_{n-2m}) \times R_{xx}(\tau_1 - \tau_2)...R_{xx}(\tau_{2m-1} - \tau_{2m}) \mathrm{d}\tau_1 ... \mathrm{d}\tau_{2m} \times x(t-\sigma_1)...x(t-\sigma_{n-2m}) \mathrm{d}\sigma_1 ... \mathrm{d}\sigma_{n-2m} \quad (5)$$

where $R_{xx}(\tau)$ is the autocorrelation function of the input at a time lag τ. Equation (5) shows that each extended Wiener kernel within a g-functional can be derived from the leading order kernel and the autocorrelation function of the input and so the bracketed term in the subscript of the kernel in Eq. (3) can be dropped.

In this paper only the first extended Wiener kernel is of interest and following a similar method to [11] can be calculated as

$$K_1(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \quad (6)$$

where $S_{yx}(\omega)$ is the cross-spectrum between output and input and $K_1(\omega)$ and $S_{xx}(\omega)$ are Fourier transforms of $k_1(\tau)$ and $R_{xx}(\tau)$ defined as

$$K_1(\omega) = \int_{-\infty}^{\infty} k_1(\tau) \mathrm{e}^{-i\omega\tau} \mathrm{d}\tau. \quad (7)$$

Using Eq. (6) the first extended Wiener kernel can be calculated provided the input spectrum and the cross-spectrum, S_{yx}, are known. These two spectra can be found either from experimental or numerical results and the time domain kernel can then be found by taking the inverse Fourier transform of Eq. (7).

3. Calculating Power from the Extended Wiener Series

The theory from the previous section can be applied in order to provide a framework for calculating the power dissipated under random excitation with a general spectrum. In order to derive a theory that encompasses as many systems as possible, this section analyses a general nonlinear N-degree-of-freedom-system identical to that of [10] of the form

$$\mathbf{M}(\mathbf{y})\ddot{\mathbf{y}} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}, t) = -\mathbf{f}(\mathbf{y})\ddot{b}(t) \quad (8)$$

where $\mathbf{y}$ is a vector of the N generalised coordinates describing the system, $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}, t)$ is a vector of length N that represents a general nonlinear dissipative and restoring force, $\ddot{b}(t)$ is a random base acceleration and $\mathbf{f}(\mathbf{y})$ is a forcing vector that relates the base motion to the force on the system.

The power, P, input by the base excitation and therefore dissipated by this system can be calculated by summing the effective force on each mass multiplied by its velocity such that

$$\begin{aligned} P &= -\mathrm{E}\left[\dot{\mathbf{y}}^{\mathrm{T}}\mathbf{f}(\mathbf{y})\ddot{b}(t)\right] \\ &= \mathrm{E}\left[z(t)\xi(t)\right] \end{aligned} \quad (9)$$

where $z(t) = \dot{\mathbf{y}}^{\mathrm{T}}\mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{y})^{\mathrm{T}}\dot{\mathbf{y}}$ and $\xi(t) = -\ddot{b}(t)$.

An extended Wiener series can be created for the random variable $z(t)$ with input $\xi(t)$ and the power from Eq. (9) can be calculated using this series as

$$\begin{aligned} P &= \mathrm{E}\left[\sum_{n=0}^{\infty} \mathbf{g}_n[k_n;\xi(t)]\xi(t)\right] \\ &= \mathrm{E}\left[k_0\xi(t)\right] + \mathrm{E}\left[\int_{-\infty}^{\infty} k_1(\tau)\xi(t-\tau)\mathrm{d}\tau\xi(t)\right] \\ &= \int_{-\infty}^{\infty} k_1(\tau)R_{\xi\xi}(\tau)\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1(\omega)S_{\xi\xi}(\omega)\mathrm{d}\omega \end{aligned} \tag{10}$$

where $S_{\xi\xi}(\omega)$ is the autocorrelation of the input ξ which has been considered as a first order Volterra functional and the orthogonality property of Eq. (4) was used to remove any g-functionals of order greater than unity.

3.1 Properties of the First Extended Wiener Kernel

In what follows, a useful property of the first kernel, namely

$$k_1(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_1(\omega)\mathrm{d}\omega = \frac{\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]}{2}, \tag{11}$$

that influences understanding of power dissipation is derived by following a similar process to convolution whereby the base excitation is considered as a series with impulses of magnitudes given by $\xi(t)$. A small change in $z(t)$ due to an impulse $\xi(T)$ at time T, termed $z(t)|_{\xi(T)}$ to differentiate it from the change in z due to previous impulses, can be assessed both from physical reasoning and from the Wiener series.

Physically, the effect of an impulse will instantaneously only change the acceleration term, $\mathbf{M}(\mathbf{y})\ddot{\mathbf{y}}$, and have no effect on the nonlinear term $\mathbf{g}(\mathbf{y},\dot{\mathbf{y}},t)$ so from Eq. (8)

$$\lim_{\delta t\to 0}\mathbf{M}(\mathbf{y})\frac{\delta\dot{\mathbf{y}}|_{\xi(T)}}{\delta t} = \mathbf{f}(\mathbf{y})\xi(T) \tag{12}$$

where $\frac{\delta\dot{\mathbf{y}}|_{\xi(T)}}{\delta t}$ represents the accelerations of the generalised coordinates only due to the excitation at time T. This equation can be described physically as a unit impulse generating a unit change in momentum.

Rearranging Eq. (12), a small change in $\dot{\mathbf{y}}$ from the excitation, $\delta\dot{\mathbf{y}}|_{\xi(T)}$, is therefore

$$\lim_{\delta t\to 0}\delta\dot{\mathbf{y}}|_{\xi(T)} = \mathbf{M}^{-1}\mathbf{f}\xi(T)\delta t. \tag{13}$$

The small time-step, δt, here is from a time just before the impulse, T^-, to a time just after the impulse, T^+, where $T^+ - T^- = \delta t$ and the magnitude of the impulse is $\xi(T)\delta t$.

The change in the velocity vector from excitation $\xi(T)$ at time T, $\delta\dot{\mathbf{y}}|_{\xi(T)}$ can also be calculated from the extended Wiener series. The change in the velocity vector provides a change in the $z(t)$ variable

$$\delta z|_{\xi(T)} = \mathbf{f}(\mathbf{y})^{\mathrm{T}}\delta\dot{\mathbf{y}}|_{\xi(T)} \tag{14}$$

where the extended Wiener series can be rewritten in a variational form

$$\delta z|_{\xi(T)} = \sum_{n=0}^{\infty}\Delta\mathbf{g}_n[k_n;\xi(T)]. \tag{15}$$

and the $\Delta\mathbf{g}_n$ terms represent the change in nth g-functional due to the excitation $\xi(T)$.

The zeroth order Δg-functional is simply zero since it does not depend on the excitation. The first order Δg-functional is

$$\begin{aligned}\Delta\mathbf{g}_1[k_1;\xi(T)] &= \lim_{\delta t\to 0}\int_{-\delta t/2}^{\delta t/2} k_1(\tau)\xi(T-\tau)\mathrm{d}\tau \\ &= \lim_{\delta t\to 0} k_1(0^+)\xi(T)\delta t \end{aligned} \tag{16}$$

because the instantaneous response to a single impulse of magnitude $\xi(T)\delta t$ is being assessed so the integral disappears. This means that immediately after the impulse the first Δg-functional contributes $k_1(0^+)\xi(T)\delta t$ to the response where the argument 0^+ in $k_1(\tau)$ represents the time instantaneously after $\tau = 0$. This is analogous to the impulse response of a linear system where immediately after an impulse, the response would be given by the value of the impulse response just after the impulse multiplied by the magnitude of the impulse. However, whilst in the linear case $k_1(\tau)$ would completely define the response, in the nonlinear case the higher order functionals may also contribute to this value so must also be assessed.

The nth Δg-functional is

$$\begin{aligned}\Delta\mathbf{g}_n[k_n;\xi(T)] = \lim_{\delta t\to 0}\sum_{m=0}^{\mathrm{floor}[(n-1)/2]} &\frac{(-1)^m n!}{(n-2m-1)!m!2^m}\xi(T)\delta t\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\times \\ &\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} k_n(\tau_1,\ldots,\tau_{2m},0^+,\sigma_2,\ldots,\sigma_{n-2m})\times \\ &R_{\xi\xi}(\tau_1-\tau_2)\ldots R_{\xi\xi}(\tau_{2m-1}-\tau_{2m})\mathrm{d}\tau_1\ldots\mathrm{d}\tau_{2m}\times \\ &\xi(T-\sigma_2)\ldots\xi(T-\sigma_{n-2m})\mathrm{d}\sigma_2\ldots\mathrm{d}\sigma_{n-2m}\end{aligned} \tag{17}$$

where the limit of the summation is different from that of Eq. (5) because only terms that involve at least one excitation term can affect δz. This term can be shown to be equal to zero by following a similar process to that of proving Eq. (5) satisfies the orthogonality condition, Eq. (4).

Combining Eqs. (13), (14), (15) and (16) and setting Eq. (17) to zero yields

$$k_1(0^+) = \mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]. \tag{18}$$

The value of the first kernel, $k_1(\tau)$, around the $\tau = 0$ point will now be explored. When $\tau = 0^-$, instantaneously before any excitation, the response must be zero because the system is causal so $k_1(0^-) = 0$. Combining with Eq. (18) suggests that at $\tau = 0$ the value of the kernel is halfway between $k_1(0^-)$ and $k_1(0^+)$, so $k_1(0) = \mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]/2$ as required by Eq. (11). This is equivalent to saying at $k_1(0)$, only half of the impulse from the excitation has occurred so the response is only half the magnitude.

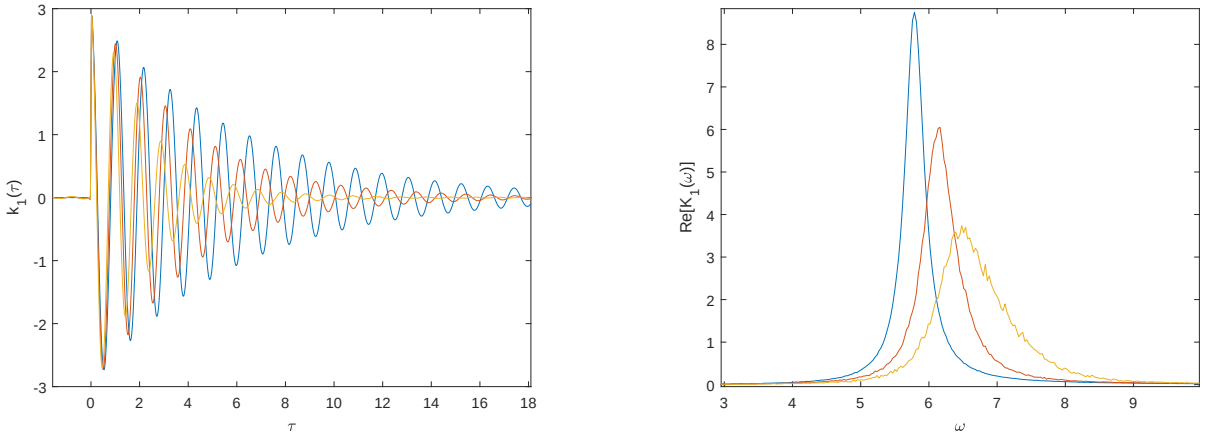
Although the triple product term $\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]$ in Eq. (18) seems physically unintuitive, it has been discussed in detail in [10] where it has been shown that for a system with a total oscillating mass, M_{Tot},

$$\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right] \leq M_{\mathrm{Tot}}. \tag{19}$$

In [10] Langley shows that if a system is constrained to reduce its number of degrees of freedom then $\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right] < M_{\mathrm{Tot}}$. However, if there is no constraint on the system, the inequality of Eq. (19) becomes an equality so $\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right] = M_{\mathrm{Tot}}$.

Since the time domain kernel, $k_1(\tau)$, is real, the frequency domain kernel, $K_1(\omega)$, is Hermitian. The imaginary part therefore does not contribute to the integral in Eqs. (10) or (11) so the equations can be modified such that the integrals are only performed over positive frequencies and the real part of the kernel giving

$$P = \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}[K_1(\omega)]S_{\xi\xi}(\omega)\mathrm{d}\omega \tag{20}$$



(a) (b)

Figure 1: First extended Wiener kernel in a) the time domain and b) real part in the frequency domain under narrowband noise when $\epsilon = 0$ (blue), $\epsilon = 0.1$ (red) and $\epsilon = 0.3$N m^{-1} (yellow).

and

$$\frac{1}{\pi}\int_0^\infty \mathrm{Re}[K_1(\omega)]\mathrm{d}\omega = \frac{\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]}{2}. \tag{21}$$

The real part of the first kernel therefore completely defines the power dissipated.

To summarise the results so far: the power dissipated by a general nonlinear oscillator, Eq. (8), under random excitation of a general spectrum, $S_{\xi\xi}(\omega)$, can be calculated using Eq. (20). To do this, the first Wiener kernel must be calculated from either simulations or experimentally using Eq. (6), but crucially for a designer of a system desiring a preliminary estimate of power dissipation, the first Wiener kernel has the property of Eq. (21).

4. Numerical Validation

In this section the results of Eqs. (20) and (21) are validated via numerical simulations of the Duffing oscillator in the form

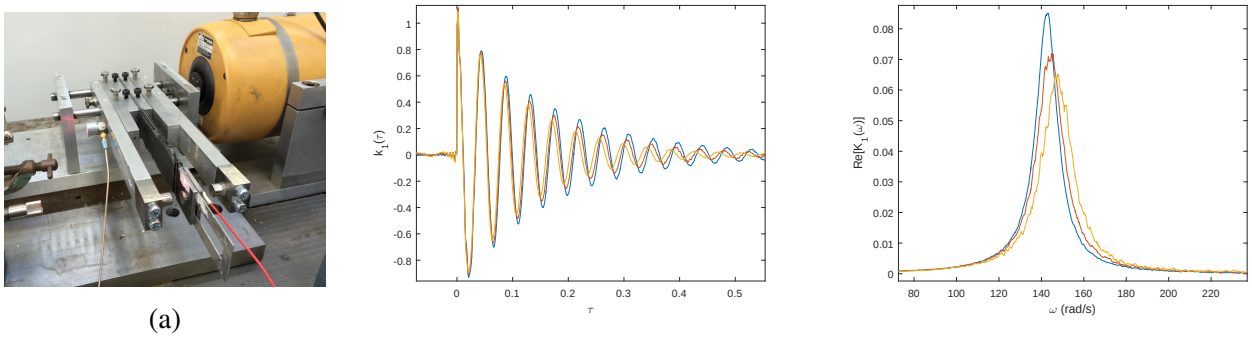
$$3\ddot{y} + \dot{y} + 100y + \epsilon y^3 = -3\ddot{b}(t). \tag{22}$$

Time domain simulations were conducted where an ensemble of results is built up by exciting a nonlinear oscillator with a number of realisations of random noise of a chosen spectrum. The ode45 routine in MATLAB was used and the first extended Wiener kernel can be found using Eq. (6). In this case $\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right] = 3$ and an ensemble of 1000 excitation time-histories with non-white spectrum

$$S_{\ddot{b}\ddot{b}}(\omega) = \frac{\pi}{2}\frac{10^4}{\left(10^2 - \omega^2\right)^2 + 36\omega^2} \tag{23}$$

was applied to three oscillators with different values of the nonlinearity constant, $\epsilon = 0$, 20 and 50N m^{-3}.

The three first kernels are plotted in both the time and frequency domains in Fig. 1 where it can clearly be seen that the initial jump of the time domain kernel is of magnitude 3. The integral of Eq. (21) is 1.53, 1.52 and 1.53kg for $\epsilon = 0$, 20 and 50N m^{-3} respectively, within 2% of the expected value of 1.5kg from the theory. The power dissipated can be calculated directly from the simulations as 8.62, 9.24 and 10.2W and from Eq. (20) as $P = 8.58$, 9.20 and 10.1W for $\epsilon = 0$, 20 and 50N m^{-3} respectively showing that the simulations agree well with the theory.



(a) (b) (c)

Figure 2: a) Experimental setup, and first extended Wiener kernel in b) the time domain and c) real part in the frequency domain when excitation magnitude is low (blue), medium (red) and high (yellow).

5. Experimental Validation

To supplement the numerical validation of Eqs. (20) and (21 in Section 4, experimental validation has been undertaken. Calculation of the power dissipation using Eq. (20) is experimentally difficult, and so only the property of the first kernel from Eq. (21) is examined. A base-excited cantilever with a tip mass as shown in Fig. 2 is used to approximate a SDOF oscillator. Neodymium cylinder magnets have been placed on the tip mass and base arms to generate a nonlinear stiffening restoring force and the tip velocity and base acceleration were found using accelerometers.

Since the experimental system is unconstrained and can be approximately modelled by a SDOF equation of motion similar to Eq. (22), the triple product, $\mathrm{E}\left[\mathbf{f}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}\right]$, is equal to the oscillating mass. The value of this mass will be difficult to calculate as some of the beam mass must be included. It is therefore preferable to modify the output of the extended Wiener series from what would be $z = m\dot{y}$ to $z = \dot{y}$ where $\dot{y}$ is the relative tip velocity. The result is that the integral over the kernel in Eq. (21) is no longer equal to $m/2$, but 1/2. This is easier to validate since no estimate of the oscillating mass is required, but is still validating the important property of the first kernel since its magnitude has been divided by its mass.

Figure 2 displays the first extended Wiener kernel in the time and frequency domains with three different excitation magnitudes where an increase in excitation increases the nonlinearity. The results show similar characteristics to the numerical results of Fig. 1 and critically, the integral over the real part of the frequency kernel behaves according to Eq. (21). The values of this integral are 0.51, 0.51 and 0.50 the low, medium and high magnitude excitation respectively providing excellent resemblance to the theory. It should be noted that any higher order resonance effects of the cantilever have a negligible effect on the integral of Eq. (21) and the spectrum of the base excitation was non-white.

6. Conclusions

A methodology has been presented for calculating the power dissipated by nonlinear MDOF systems under general random base excitation. The Wiener series approach for a Gaussian random input with general spectrum has been applied and the power dissipated is shown to be dependent only on the first kernel and the spectrum of the input according to Eq. (20). Calculation of the first kernel can be made either via simulations or from experimental data and it is shown to have the property that the integral over the frequency domain kernel is proportional to the total oscillating mass, Eq. (21), for an unconstrained system. Both numerical and experimental validation has been undertaken and produced strong results when compared to the values predicted by theory.

The combination of Eqs. (20) and (21) provides a simple conceptual understanding of power

flow in nonlinear randomly excited systems. The first extended Wiener kernel displays peaks around the resonances of the system therefore for applications where power input to the system is to be minimised, the resonances of the system should be designed to be at a frequency where the input spectrum is low. Conversely, for applications such as energy harvesting where power is to be maximised, the resonances of the system should be tuned to frequencies where the input spectrum is high. Whilst these conclusions are conceptually obvious, the derivation of Eq. (20) provides a rigorous mathematical framework for calculating power and Eq. (21) provides valuable information concerning how much power can be dissipated.

Acknowledgements

The authors would like to thank the EPSRC for providing the Doctoral Training Award and Programme Grant "Engineering Nonlinearity" EP/K003836/1 that have funded this research.

REFERENCES

1. Beeby, S. P., Tudor, M. J. and White, N. M. Energy harvesting vibration sources for microsystems applications, *Measurement Science and Technology*, **17** (12), R175–R195, (2006).

2. Daqaq, M. F. Response of uni-modal duffing-type harvesters to random forced excitations, *Journal of Sound and Vibration*, **329** (18), 3621–3631, (2010).

3. Daqaq, M. F. Transduction of a bistable inductive generator driven by white and exponentially correlated Gaussian noise, *Journal of Sound and Vibration*, **330** (11), 2554–2564, (2011).

4. Daqaq, M. F. On intentional introduction of stiffness nonlinearities for energy harvesting under white Gaussian excitations, *Nonlinear Dynamics*, **69** (3), 1063–1079, (2012).

5. Halvorsen, E. Energy Harvesters Driven by Broadband Random Vibrations, *Journal of Microelectromechanical Systems*, **17** (5), 1061–1071, (2008).

6. Scruggs, J. An optimal stochastic control theory for distributed energy harvesting networks, *Journal of Sound and Vibration*, **320**, 707–725, (2009).

7. Green, P., Worden, K., Atallah, K. and Sims, N. The benefits of Duffing-type nonlinearities and electrical optimisation of a mono-stable energy harvester under white Gaussian excitations, *Journal of Sound and Vibration*, **331** (20), 4504–4517, (2012).

8. Halvorsen, E. Fundamental issues in nonlinear wideband-vibration energy harvesting, *Physical Review E*, **87** (4), 042129, (2013).

9. Langley, R. S. A general mass law for broadband energy harvesting, *Journal of Sound and Vibration*, **333** (3), 927–936, (2014).

10. Langley, R. S. Bounds on the vibrational energy that can be harvested from random base motion, *Journal of Sound and Vibration*, **339**, 247–261, (2015).

11. Schetzen, M., *The Volterra and Wiener theories of nonlinear systems*, A Wiley - Interscience publication, Wiley (1980).

12. Westwick, D., Kearney, R., in Medicine, I. E. and Society, B., *Identification of Nonlinear Physiological Systems*, Biomedical Engineering, Wiley (2003).

13. Lee, Y. W. and Schetzen, M. Measurement of the Wiener Kernels of a Non-linear System by Cross-correlation, *International Journal of Control*, **2** (3), 237–254, (1965).