TRANSMISSION AND SCATTERING BY DOUBLY PERIODIC STRUCTURES

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1. INTRODUCTION

Steady state acoustic scattering is of fundamental importance to SONAR. The ability to predict acoustic quantities plays an important part in the design and acceptance of underwater systems. Problems involving large scale structures present difficulties. In principle it is possible to consider monochromatic scattering using precise techniques such as a coupled finite element and boundary element approach [1]. However apart from at low frequencies these methods can be expensive computationally. For certain extended structures such as the sea bed even this is not possible. If the structure has some exploitable symmetry then the effort can be reduced. Here we consider structures that are many wavelengths in extent which exhibit a high degree of spatial repetition. It becomes convenient to assume that the object is of effectively infinite extent. Thus the structure is idealised as an acoustic baffle with well defined transmission and reflection characteristics.

2. THEORY

A monochromatic plane wave, P_{inc} of form, $P_0 e^{i(\omega t - k.x)}$, in a compressible fluid with no mean flow is incident on a planar structure composed of a doubly periodic plane (figure 1.). The system conveniently may be composed of three regions; the fluid above and below the structure and the structure itself (regions *I,III* and *II* respectively, figure 1.). The pressure distribution in the fluid regions satisfies the Helmholtz wave equation. The incident wave in the fluid region *I* produces reflected and, depending on the nature of the structure, transmitted wave components into region *III* beneath. Higher scattering order waves may be excited which either decay with distance from the plane or propagate to the far field. Associated with each wave is a wavenumber, k_{rs}, which if real refers to a propagating wave, and a complex amplitude, which depending on fluid region is either a transmission, T_{rs}, or a reflection coefficient, R_{rs}.

Owing to the assumed structural periodicity it is natural to invoke Bloch's theorem. This is the mathematical embodiment of the spatial periodicity in the problem and results in considering the two dimensional infinite spatial Fourier expansions of the acoustic and displacement fields. The former may be written for the two fluid regions as,

$$P_I(x,y,z) = P_{inc} + \sum_{rs} R_{rs} e^{i(\alpha_r x + \beta_s y + k_{rs} z)}$$
$$P_{III}(x,y,z) = \sum_{rs} T_{rs} e^{i(x\alpha_r + y\beta_s - zk_{rs})} \qquad (1)$$

Here, α_r and β_s are spatial wavenumbers of each wave component projected on to the x and y axes. The direction of propagation of any wave is simply (α_r, β_s, $\pm k_{rs}$) depending on fluid region. Similar relations apply to the normal velocities at each boundary. A fuller exposition of the theory is available as an addendum to [2].

3. NUMERICAL DISCRETISATION

Bloch's theorem requires that only a single unit cell need be considered (figure 1). This includes also small regions of fluid above and below the structure to accurately describe the acoustic near field. The PAFEC finite element analysis system [2] has been adapted to model both the fluid regions and the structure. The Helmholtz equation was discretised using a formulation of the finite element method as described in [3] using quadratic shape functions. The structure was idealised with a mesh of standard isoparametric three dimensional elastic elements [3].

Assuming harmonic motion of circular frequency, ω, results in a set of linear equations,

$$\begin{bmatrix} \{[S]-\omega^2[M]\} & [C]^T \\ [C] & \{\omega^{-2}[S]_a-[M]_a\} \end{bmatrix} \begin{bmatrix} [u] \\ [p] \end{bmatrix} = \begin{bmatrix} [F] \\ [F]_a \end{bmatrix} \quad , \qquad (2)$$

to be solved, where, $[u]$, is a vector of displacements on the structural mesh, $[p]$, is a vector of pressures on the acoustic mesh. $[S]$ and $[M]$ are structural stiffness and mass matrices respectively while those with a subscript, a, pertain to the acoustic fluid regions. $[F]$ is a vector representing any external structural forcing and $[F]_a$ is an equivalent set of acoustic "forces" acting at the top and bottom fluid surfaces S^+ and S^- (figure 1) caused by the imposition of surface vibration due to the incident pressure field. Coupling matrices $[C]$ are required to provide consistent continuity relations between the fluid and the structure. Equation (2) is reduced by imposing complex generalised constraints ensuring that the pressure and displacement fields obey the correct periodic boundary conditions.

The acoustic "forces" need careful consideration and are determined by the impedances of the spatial Fourier coefficients for the expansions of the acoustic pressure fields in regions *I* and *III*. The spatial expansions resulting in transforms of the "forces" to the pressure fields are truncated to $2N_x+1$ and $2N_y+1$ terms respectively. This yields the system matrices,

$$\begin{bmatrix} \{[S]-\omega^2[M]\} & [C]^T \\ [C] & \{\omega^{-2}[S]_a-[M]_a-[X]\} \end{bmatrix} \begin{bmatrix} [u] \\ [p] \end{bmatrix} = \begin{bmatrix} [F] \\ [P] \end{bmatrix} \quad , \qquad (3)$$

where $[P]$, is the equivalent "forces" due solely to the incident wave and $[X]$ is the transform matrix with of the order of $2N_x+1$ by $2N_y+1$ rows. The size of $[X]$ is governed by the number of possible evanescent or propagating acoustic waves present in the problem and the number of acoustic degrees of freedom retained after reduction. Clearly a sufficient number of waves must be maintained for a good representation of the acoustic fields. The matrix equation (3) is solved and the transmission and reflection coefficients derived via an implicit inverse transform.

4. TEST PROBLEMS

Two problems are addressed. The first tests simple acoustic reflection and refraction with an actual structure while the second considers higher scattering orders on an idealised corrugated boundary. The fluid in each case is water with a sound speed of 1500 ms^{-1} and density of 1000 kgm^{-1}.

An infinite isotropic elastic plate [4] composed of a polyurethane material (Young's modulus = 4.1×10^9 Pa, Poisson's ratio = 0.4 and density = 1.4×10^3 kgm^{-3}) is isonified by a plane wave incident at $\theta_0 = 10°$ from the normal coplanar with the *xz* plane. The structural material may exhibit intrinsic damping by imposing complex moduli. The material's bulk wave speed is similar to that of the surrounding fluid while its specific acoustic impedance is double that of water. The plate is nearly acoustically transparent. This case was chosen by Hennion et al. [4] as a test of their finite element method. In contrast to this work, their method was restricted to two dimensional problems only. This has recently been extended to fully three dimensional problems [5] from which some of this work is derived.

Transmission coefficients are evaluated over the frequency range 100 - 10^4 Hz. The infinitesimal degree of symmetry forbids all but a single transmitted wave and hence only the minimum number of Fourier coefficients is required ($N_x = 0$, $N_y = 0$). This results in one wave in total. The simple mesh comprises some 24 elements (16 structural) the mesh density determined by the highest frequency considered and the sound speed in the material.

Heaps [6] obtained a closed form solution to the problem of two-dimensional plane wave scattering by a pressure free surface with a sinusoidal spatial profile (of form $A\cos(2\pi x/L)$). His derivations for the various reflection coefficients are given via a recurrence relation. These expressions are in the form of power series expansion of a small parameter, $\varepsilon = kA$, essentially the ratio of the acoustic wavelength to the amplitude of the profile. Heaps cites explicit formulae ignoring parameter terms of forth order or higher. Heaps' work is generalised to consider point sources in the fluid. This capability is not within the finite element method at present and that part of [6] is ignored in this paper.

Heaps' [6] theory requires that the profile amplitude be small compared to the spatial wavelength. We consider a profile with $A = 0.5$ m and $L = 8.0$ m. The mesh is entirely of acoustic elements (70 in all). The somewhat large number is more a matter of attempting to reproduce the correct profile with piecewise quadratic functions rather than considerations as to frequency. The set of Fourier components ($N_x = 4$, $N_y = 0$) results in a total of 9 waves considered. Two separate sets of calculations are presented. The first considers fixed angle of incidence (10° off- normal) and varying the frequency over the range of 100 - 400 Hz. The second at a fixed frequency of 200 Hz. has varying angle of incidence, θ_0, from 0° (normal) to 80°.

5. RESULTS and DISCUSSION

The magnitude of the frequency dependent transmission coefficient for the elastic plate derived by the finite element scheme (this work) shows excellent accord with an expression derived exactly (see figure 2.). This is especially so at low frequencies below the first coincidence resonance (the position of which is well predicted by this work to be at 4.4 kHz). At higher frequencies this work slightly over predicts transmission by up to 2%. This may be a consequence of the coarse mesh which results in a slightly over stiff plate. This is born out by the second resonance predicted by this work being a little above that of exact theory (8.39 cf. 8.36 kHz.).The level of accord between this work and the exact solution is similar to the two dimensional finite element results of [4].

The magnitudes of the reflection coefficients for the sinusoidal pressure release surface predicted by this work are tabulated (see, tables 1. and 2.) with those of Heaps [6]. The reflected wave amplitudes follow Heaps' notation and only propagating amplitudes are shown (a null entry corresponds to an evanescent component). The non-specular angles in [6] are given by the exact relation, $\sin\theta_n = \sin\theta_0 + n\lambda/L$ (e.g. $\theta_{+1} = 40°$, at 400 Hz.). Clearly there is good accord as to the conditions for the on-set of such non-specular propagation. The finite element method predicts the same number of propagating waves as does Heaps. For the specific case of fixed angle of incidence (table 1.) the PAFEC amplitudes (marked by *) agree well with Heaps' values particularly at the low frequencies where the Heaps' expansion should be good. Here the agreement is typically within 2%. At the highest frequency considered, this work differs from the Heaps' approximation by less than 15%. Here the convergence of the Heaps series solution is in doubt with the smallness parameter, ε^4, being ~ 0.5.

For differing angles of incidence again a similar level of agreement with Heaps [6] and this work is seen (table 2.). The on-set of excitation of the second order wave (A_{+2}) at just less than 70° directly back towards the acoustic source of the incident plane wave is well reproduced. However there is some disparity in the magnitude of the specular amplitude, $|A_0|$, particularly at high angles of incidence (table 2.). The Heaps' prediction decreases monotonically between 10° and 50° while the finite element amplitude rises after 30° to around unity at 50°. This may be due to higher order terms neglected by Heaps [6] becoming significant at near grazing incidence. Here the surface is effectively "shiny" and a very large (almost unity) specular component is expected. Given that the formulae of Heaps [6] are approximate it is remarkable that they produce such good level of accord with this work. No exact solution to this problem exists but it is gratifying that the finite element results are consistent with the truncated series solution [6].

6. CONCLUSIONS

The finite element method can be used to predict the reflection and transmission characteristics of a spatially periodic scatterer of effectively infinite extent. Higher non-specular waves are included in this treatment which is exact within the limits of the finite element approximation. The reflection and transmission wave amplitudes are obtained as are their directions. It is important to remember that the derived beams are plane waves with no "width"; a consequence of considering an infinitely large structure and Heisenberg's Uncertainty Principle!

The use of the finite element method allows for great freedom in structural modelling. All kinds of complicated structures can be considered provided a degree of spatial regularity exists. Incorporating structural losses is conceptually simple by adopting complex elastic moduli.

7. REFERENCES

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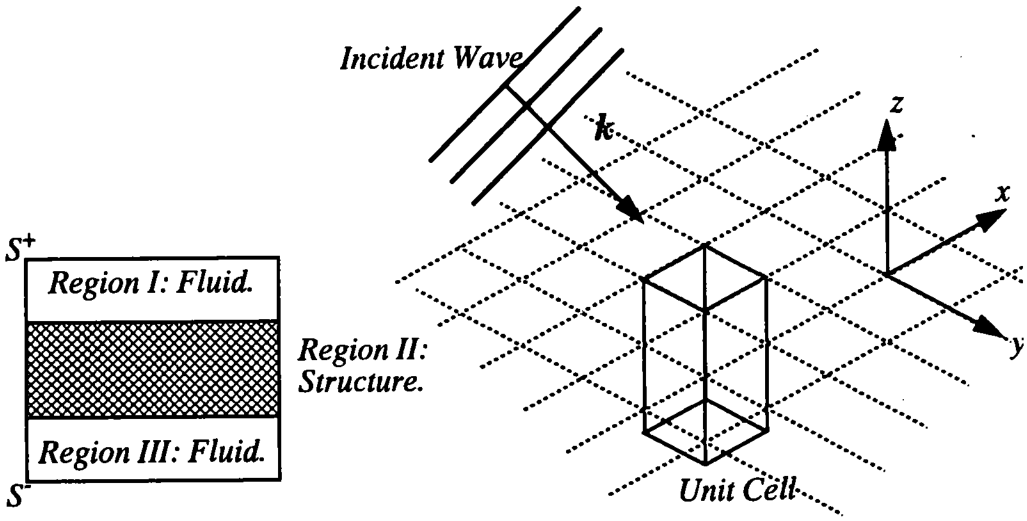
frequency	$\|A_0\|$	$\|A_0^*\|$	$\|A_{-1}\|$	$\|A_{-1}^*\|$	$\|A_{+1}\|$	$\|A_{+1}^*\|$	$\|A_{-2}\|$	$\|A_{-2}^*\|$
100 Hz.	1.0021	1.0039						
150 Hz.	1.0024	1.0301						
200 Hz.	0.9443	0.9446	0.3976	0.3937				
250 Hz.	0.8354	0.8407	0.4626	0.5152	0.4499	0.4584		
300 Hz.	0.7051	0.7286	0.5146	0.5786	0.5075	0.5656		
350 Hz.	0.5602	0.5900	0.5698	0.6560	0.5400	0.6165	0.2466	0.2150
400 Hz.	0.3958	0.4605	0.6115	0.6848	0.5805	0.6310	0.3302	0.2935

Table 1: Reflection coefficients for specular and higher scattering orders for a plane wave incident on pressure release sinusoidal corrugations for various frequencies and fixed angle of incidence, θ =10° (this work using the PAFEC finite element code).*

θ	$\|A_0\|$	$\|A_0^*\|$	$\|A_{-1}\|$	$\|A_{-1}^*\|$	$\|A_{-2}\|$	$\|A_{-2}^*\|$
0°	0.9389	0.9409	0.4032	0.4061		
10°	0.9443	0.9446	0.3946	0.3937		
20°	0.9322	0.9417	0.3664	0.3862		
30°	0.9256	0.9329	0.3267	0.3541		
40°	0.9224	0.9634	0.2782	0.3147		
50°	0.9215	1.0135	0.2232	0.2568		
60°	0.9219	1.0407	0.1630	0.2020		
70°	0.9225	0.9696	0.1084	0.1388	0.0300	0.0288
80°	0.9237	0.9694	0.0544	0.0694	0.0152	0.0143

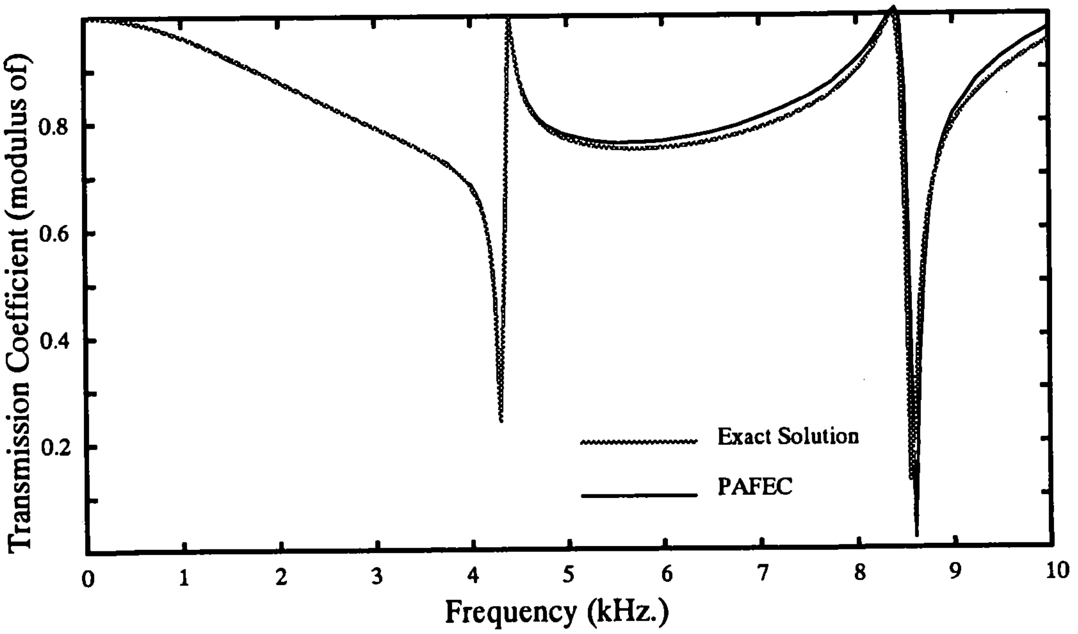
Table 2: Reflection coefficients for specular and higher scattering orders for a plane wave incident on pressure release sinusoidal corrugations at a fixed frequency (200 Hz.) and varying angle of incidence (this work using the PAFEC finite element code).*

figure 1.



Plane wave incident on periodic structure illustrating the unit cell.

figure 2.



Transmission coefficient for a polyurethane plate at fixed angle of incidence (θ = 10°).