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EIGENVECTOR ROTATION AND ITS APPLICATION TO HIGH RESOLUTION SPECTRUM ESTIMATION

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INTRODUCTION

There is currently a great deal of interest in adaptive methods which are able to resolve closely spaced acoustic sources incident on an array. A large number of algorithms have been proposed, ranging from Minimum Variance (or Maximum Likelihood Method (MLM), and Maximum Entropy to eigenvector methods. It is generally accepted that eigenvector methods have good resolution properties, and there is currently a great deal of research on these methods.

Most eigenvector methods are based on the orthogonality between the noise subspace and the incident direction vectors. This orthogonality is only an asymptotic property, but nevertheless most eigenvector methods do have good resolution properties even when a relatively small number of time samples are available. The most popular eigenvector method is the MUSIC method (MUltiple SIgnal Classification) proposed by Schmidt [1] and Bienvenu and Kopp [2]. Other methods have also been proposed, and the Kumaresan and Tufts (KT) [3] algorithm does have good resolution properties.

The present paper is concerned with using the source subspace. Asymptotically, this subspace will contain all of the information about the signals. Although this is useful, it is not particularly easy to make use of this fact. The difficulty is that individual eigenvectors align themselves to be orthogonal, and so the direction vectors will not, in general, be equal to the eigenvectors. However, it has been shown [4] that the eigenvectors are related to the signal directions by a unitary matrix, and it is this property which will be used in this paper.

The direction vectors may be obtained by rotating weighted eigenvectors, but unfortunately there are an infinite number of possible rotation matrices to choose from. In order to identify the direction vectors correctly, it is necessary to specify additional prior information. In this paper, the additional information is supplied by the Varimax criterion [5]. This criterion was originally used to interpret psychological data, and as the name suggests, the criterion is normally maximised. In the present approach, the same criterion is used but the function is minimised.

EIGENVECTOR ANALYSIS

Consider an arbitrary array, having p sensors, with R sources incident on the array at one particular frequency, w. The data from each sensor is passed through a FFT processor and the complex valued frequency output for frequency, w, forms an element of the data vector $\underline{y}$:-

$$\underline{y} = \sum_{j=1}^{k} x_j \underline{h}_j + \underline{n} \qquad (1)$$

Let the k incident signals, x_j, have power σ^2_j and have direction vectors, $\underline{h}_j$.

Since this is a frequency domain representation, the direction vectors will consist of complex exponentials with exponent proportional to the time delay suffered by signal, x_i, for a particular sensor. The vector, $\underline{n}$, represents additional noise caused either by sensor inaccuracy or the background ocean environment.

Consecutive vector FFT outputs, $\underline{y}_1, \underline{y}_2 \ldots$, may be used to form a sample covariance matrix.

$$R = \frac{1}{M} YY^{\dagger} \tag{2}$$

where $Y = [\underline{y}_1, \underline{y}_2 \ldots \underline{y}_M]$

Asymptotically (i.e. for large M), R may be written as:

$$R = \sigma^2 R_o + DD^{\dagger}$$

where R_o is the background noise correlation matrix and D is a (p×k) direction matrix defined as:

$$D = [\sigma_1 \underline{h}_1, \sigma_2 \underline{h}_2 \ldots \sigma_k \underline{h}_h] = [\underline{d}_1 \underline{d}_2 \ldots d_k]$$

R may also be represented by an eigenvector decomposition

$$R = \sigma^2 R_o + R_o^{\frac{1}{2}} \Lambda \Lambda^{\dagger} R_o^{\frac{1}{2}\dagger} \tag{3}$$

where Λ is a (p×p) matrix of weighted eigenvectors of $R_o^{-\frac{1}{2}\dagger} R R_o^{-\frac{1}{2}}$:

$$\Lambda = [\sqrt{\lambda_1 - \sigma^2}\ \underline{\xi}_1;\ \sqrt{\lambda_2 - \sigma^2}\ \underline{\xi}_2 \ldots \sqrt{\lambda_k - \sigma^2}\ \underline{\xi}_k]$$

and $R_o^{\frac{1}{2}}$ is a Cholesky factor of R_o.

In general, the direction matrix, D, will be related to the eigenvector matrix Λ, by a square (k×k) matrix B:

$$D = R_o^{\frac{1}{2}} \Lambda B \tag{4}$$

where B satisfies $BB^{\dagger} = I$. For the case, k=2; it is shown in [4] that B may take the form

$$B = \begin{bmatrix} \cos\theta, & -\sin\theta e^{j\delta} \\ \sin\theta e^{-j\delta}, & \cos\theta \end{bmatrix} \tag{5}$$

The matrix $R_o^{\frac{1}{2}}\Lambda$ is easy to obtain, but difficult to interpret. The direction matrix, D, is easy to interpret; but cannot be found unless we supply additional prior information. If we could specify B with the aid of additional information, this would enable D to be estimated.

The MUSIC algorithm estimates signal directions by seeking directions (θ) which maximise:

$$f(\theta) = \frac{1}{\sum_{i=k+1}^{p} \left[\underline{h}^{\dagger}(\theta)R_o^{-\frac{1}{2}\dagger}\underline{\xi}_i\right]^2} \qquad (6)$$

This algorithm supplies the necessary additional information by specifying the expected family of direction vectors $\underline{h}(\theta)$. It is easy to show that maximising $f(\theta)$ is exactly equivalent to maximising $g(\theta)$:

$$g(\theta) = \underline{h}^{\dagger}(\theta)R_o^{-\frac{1}{2}\dagger}\Lambda_s\Lambda_s^{\dagger} R_o^{-\frac{1}{2}}\underline{h}(\theta) \qquad (7)$$

where $\Lambda_s = [\underline{\xi}_1; \underline{\xi}_2 \cdots \underline{\xi}_k]$

This is the Bartlett Principle component method [6]. It is interesting to note that $g(\theta)$ may be rewritten

$$g(\theta) = \left(\underline{h}^{\dagger}(\theta)R_o^{-\frac{1}{2}\dagger}\Lambda_s B\right)\left(B^{\dagger}\Lambda_s^{\dagger}R_o^{-\frac{1}{2}}\underline{h}(\theta)\right)$$

$$= \underline{h}^{\dagger}(\theta)D_1D_1^{\dagger}\underline{h}(\theta) = \sum_{i=1}^{k} g_i^2(\theta) \qquad (8)$$

Note that D_1 is closely related (but not equal) to the matrix D. The Bartlett method (and hence MUSIC also) may be seen to average the responses of all of the source eigenvectors. By comparison, if the matrix B could be found, it would be possible to find the responses, $g_i^2(\theta)$, of individual direction vectors in order to interpret the signal directions.

PRIOR INFORMATION

It has been seen that the MUSIC method makes use of additional information contained in the family of possible direction vectors. An alternative method of including additional prior information is to postulate many different rotation matrices and reject direction vector estimates which are not consistent with the prior information.

It has already been stated that the direction vectors will consist of complex exponentials representing the time delays for particular incident signals. One of the properties of such exponentials is that the moduli of all the terms in a given direction vector should be identical. This would be exactly true asymptotically and will be approximately true in the small sample case.

If the moduli of the individual direction vectors are equal, the quantity ϕ defined by:

$$\phi = \sum_{i=1}^{k}\sum_{j=1}^{p} (|d_{ji}|^2 - \bar{d}_i)^2 \qquad (9)$$

$$\text{where } \bar{d}_i = \frac{1}{p} \sum_{j=1}^{p} |d_{ji}|^2$$

will take the value zero. This is the Varimax criterion [5] and by minimising the quantity we are seeking vectors, $\underline{d}_i$, which approximate complex exponentials.

For the two signal case, the rotation matrix B only depends on two parameters, θ and δ. These parameters may be varied until the Varimax criterion is minimised. When the criterion has reached a minimum, it is assumed that the correct rotation matrix has been found.

DIRECTION ESTIMATION

Once the rotation matrix has been estimated, it may be used to calculate direction vectors, $\underline{d}_i$. The directions corresponding to these vectors may be estimated in a number of ways.

One direct way of estimating the signal direction is to seek a θ that maximises

$$\beta_j(\theta) = |\underline{h}^{\dagger}(\theta)\underline{d}_j|^2 \qquad (10)$$

In general, this function will have a global maximum in the signal direction and local maxima in many other directions. This technique is reliable but is liable to be very expensive since the maximisations need to be performed for all of the k signal direction vectors.

An alternative way of estimating signal directions is to attempt to interpret the phase information contained in the direction vectors. This may be achieved by taking logarithms of the elements in vectors $\underline{d}_i$. For example consider that the array is linear with equispaced sensors (spacing = x). Then $\underline{d}_i$ will have the form

$$\underline{d}_i^{\dagger} = \sigma_i e^{j\phi i} \left[1, e^{j\alpha_i}, e^{j2\alpha_i}, \ldots, e^{j(p-1)\alpha_i}\right]$$

$$\text{where } \alpha_i = \frac{2\pi x \cos}{\lambda} \theta_i$$

Taking logarithms of the individual terms, gives imaginary parts of the form

$$[\phi_i, \phi_i+\alpha_i, \phi_i+2\alpha_i \ \ldots \ ]$$

which may be plotted as a straight line to obtain the slope α_i. However, before the points may be plotted, it is necessary to restore the multiples of 2π which are lost in the circular phase representation. This process is called phase unwrapping [7]. The authors believe that the phase unwrapping task is simpler than seeking a global maximum of a multimodal function.

The above example was for a linear equispaced array, but the technique may also be applied to other array shapes, although the phase unwrapping is easier to accomplish if the array has at least some portions which are straight or nearly straight.

RESULTS

This section demonstrates the techniques described in this paper by considering a number of examples. One of the most important properties of the rotation method is its ability to resolve closely separated signals. The following simulation was implemented to show this.

A ten element equispaced line array with omidirectional sensors, at a frequency corresponding to an element separation of $\lambda/5$, was assumed. Two plane wave signals were modelled with additive uncorrelated sensor noise. The signals have wavenumbers of 0.12 and 0.16 with power of 0dB and -3dB, the background noise power is 0dB. Sampled data sets were then formed for this situation, each set of data was obtained from one hundred sets of random data. Even though the array is equispaced no Toeplitz assumptions are made so the results should be similar to those expected when using non-equispaced arrays.

Figure 1 shows the MUSIC spectra for the first ten samples. It is clearly very difficult to locate the signal direction. The same ten samples are used in Figure 2 where the KT algorithm [6] has been implemented. These spectra exhibit superior resolution to the MUSIC spectra. One hundred sampled wavenumber estimates after rotation and phase unwrapping are shown in Figure 3. The two signals have been resolved, and are in approximately the correct directions. The first ten direction estimates compare well with those obtained using the MUSIC and KT methods.

An important feature of rotation that has not been discussed in this paper, is its ability to perform well in the presence of phase and amplitude errors. This was addressed in a previous paper [8] and the performance of rotation was shown to be very superior to both the MUSIC and KT techniques.

The evaluation of the correct rotation parameters requires the minimisation of the function described in equation (9). For the two signal case it is possible to plot this function. In this simulation we assume two signals at wavenumbers 0.12 and 0.16 with a background of power 0dB. Figure 4 shows a typical plot of $1/\phi$ against θ and δ from a set of sampled data with both signals having power of 3dB. It is very easy to detect the large maximum. If the powers of the signals are reduced to -10dB, the plot of $1/\phi$ presented in Figure 5, demonstrates that at this signal to noise ratio the function is very difficult to optimise.

Although the phase unwrapping algorithm implemented in this paper is sub-optimal, it is very quick and its performance has been acceptable. The alternative to phase unwrapping is to locate the global maximum of the function in equation (10). Figure 6 shows two beam patterns obtained from sampled data, for signals at wavenumbers 0.12 and 0.16, having powers of -3dB with a background of 0dB. It would be difficult and expensive to perform a global maximisation of this function. At this signal to noise ratio the phase unwrapping algorithm performs very well. For the same sample used in Figure 6, the wrapped phase for one of the signal directions is shown in Figure 7. After phase unwrapping a least squares fit may be used to obtain a signal direction estimate, this is illustrated in Figure 8.

CONCLUSIONS

It has been demonstrated that source eigenvectors may be rotated to yield the incident direction vectors. This rotation may be accomplished using a criterion which gives preference to vectors with constant modulus. Simulation results have demonstrated the effectiveness of the technique.

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FIG. 1 : 10 MUSIC spectra

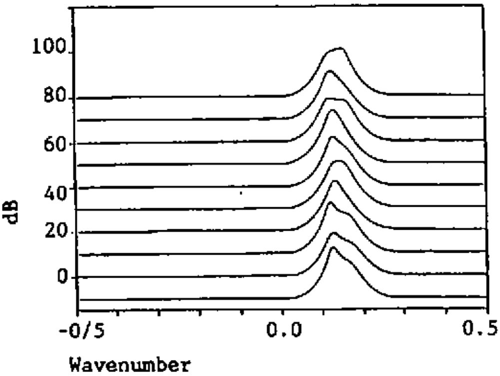


FIG. 2: 10 KT spectra

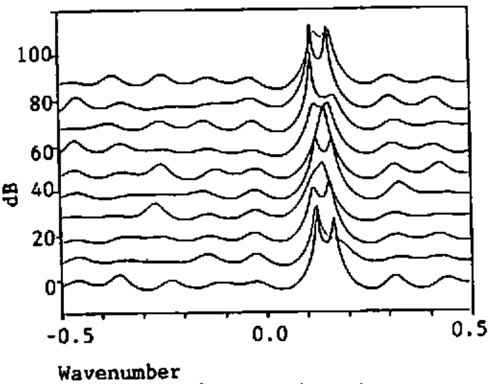
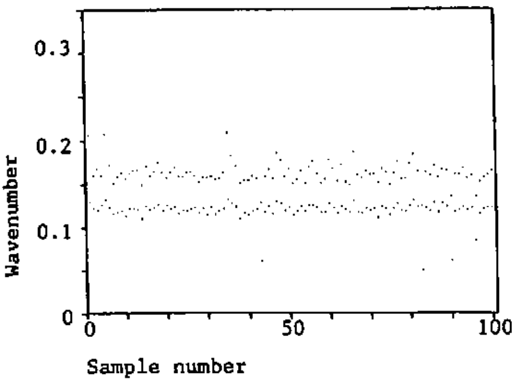


FIG. 3 : 100 Rotated estimates



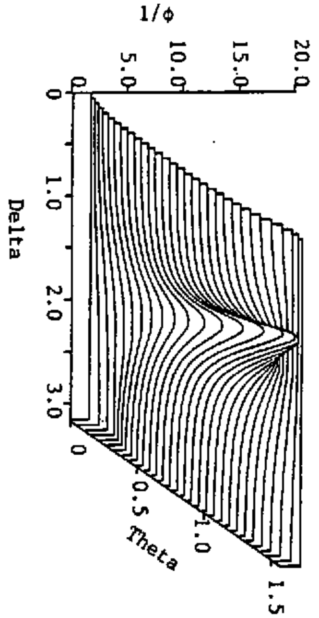


FIG. 4 : 1/ϕ, SNR = 3dB

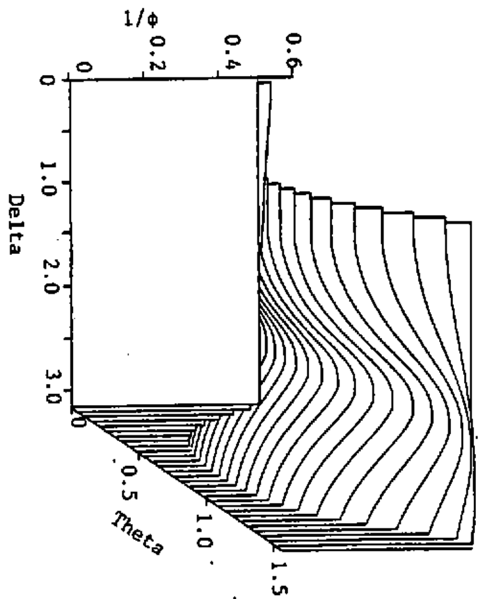


FIG. 5 : 1/ϕ, SNR = -10dB

FIG. 6 : 2 Beam patterns

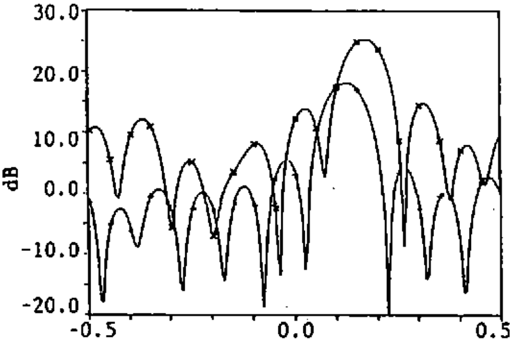


FIG. 7 : Wrapped phase

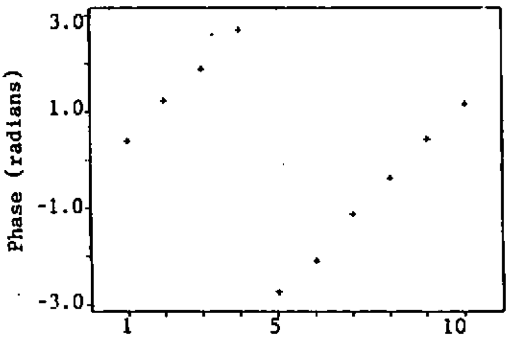


FIG. 8 : Unwrapped phase