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HIGH RESOLUTION BEAMFORMING

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1. INTRODUCTION

The MUSIC algorithm was proposed 10 years ago. Since then, numerous authors have proposed algorithms which are claimed to be superior in some way. However, in spite of all this research, the MUSIC algorithm is still generally regarded as the most effective for arrays with arbitrary geometry. Later methods have generally not been accepted because of the following reasons.

(1) Additional prior information has been assumed. A common example of this is a Toeplitz assumption for linear equispaced arrays. In addition to the geometric assumption, this assumes that there is no multipath present. Consequently, any method based on this assumption will perform very badly if the signals are correlated.

(2) The array shape is restricted in some way. An example of this, is the linear prediction methods which assume linear equispaced arrays. Although these methods are useful, many arrays do not satisfy this requirement.

(3) Very costly iterations are required. These methods make a general purpose processor very expensive to implement, and there is sometimes a question about convergence. Examples of these methods are Maximum Likelihood methods, which generally have good performance, but are often very expensive to implement.

(4) Some methods are notoriously sensitive to modelling errors. The MUSIC algorithm is not particularly good on this point either, but some methods are much worse. These errors are very likely to occur in many designs, and so this is a very important feature.

(5) The noise statistics are assumed to be known exactly, or the noise is assumed to be uncorrelated. For many applications, these assumptions are false, leading to severe degradation of performance.

(6) Some algorithms give performance which is very similar to MUSIC, and so there is no <u>real</u> benefit. In order to be worth accepting, any method needs to have either a significantly lower noise threshold, or a lower variance (and preferably both).

The present paper presents yet another method, but the author believes that this algorithm could replace MUSIC, since the method makes no additional assumptions, is not iterative, and gives a <u>substantial</u> improvement in the noise threshold (≈10dB).

We will assume narrow-band processing, but not make any assumptions about array shape. There should be no difficulty in extending the method to the broad-band case. For convenience we will use a linear, equispaced array for the simulations. With this type of array the direction vectors will be of the form:

$$\underline{h}^{\dagger}=[1,\exp(-\varphi),\exp(-2\varphi),\ldots,\exp(-(p-1)\varphi)] \qquad (1)$$

where

$$\varphi = 2\pi j d\cos(\theta)/\lambda$$

with element spacing (d),wavelength (λ), and incident angle (θ). In some cases we will use several (k) direction vectors simultaneously, and use the notation.

$$H = [\ \underline{h}_1\ \underline{h}_2\ \cdots\ \underline{h}_k\ ] \tag{2}$$

The task is to identify the directions of arrival, when the data $\underline{y}_i$ has been formed from:

$$\underline{y}_i = H\ \underline{x}_i + \underline{n}_i$$

where $\underline{x}_i$ are the incident signals, and $\underline{n}_i$ is a noise vector.

2. BASIC PRINCIPLE

Consider a p-element array which has received N snapshots of data. We may form a p x N data matrix, Y.

$$Y = [\ \underline{y}_1\ \underline{y}_2\ \underline{y}_3\ \cdots\ \underline{y}_N]\ /\sqrt{N} \tag{3}$$

Reilly and Wong [1] have used Bayesian analysis to show that if the noise covariance matrix is unknown, the correct direction vector matrix, H, minimises:

$$\det_0[(I - H\ (H^\dagger H)^{-1}\ H^\dagger)R(I - H\ (H^\dagger H)^{-1}\ H^\dagger)\ ] \tag{4}$$

where $R = Y\ Y^\dagger$, and $\det_0$ denotes the product of all the non-zero eigenvalues. The important property is that no knowledge of the noise covariance is required, or assumed. Unfortunately, this algorithm is extremely expensive. The approach taken in this paper has many similarities with Equation (4), but at greatly reduced cost.

If there is no noise and there are k signals present, the rank of the matrix ,Y, will be k (assuming $N \geq k$ and $p \geq k$). With uncorrelated noise present the first k singular values should be larger than the remaining singular values (min(p-k,N-k)), which should have singular values approximately equal to the standard deviation of the noise.

A rank of k for the noise-free case states that the matrix may be written in terms of k independent components. Alternatively, we may express this in terms of the covariance matrix, R. With k signals, these components may be described solely in terms of k eigenvalues and vectors, so that

$$R = U\ \Sigma\ U^\dagger \tag{5}$$

where Σ is a diagonal matrix of eigenvalues,λ_i, and the columns of U contain the eigenvectors of R. The signal components may be described by

$$R_S = U_S \Sigma_S U_S^\dagger \tag{6}$$

where Σ_s and U_s contain the first k eigenvalues and eigenvectors of R.

Now consider the eigenvalues of the matrix (c.f. Eqn. (4))

$$R_r = [\ (I - H\ (H^\dagger H)^{-1}\ H^\dagger)R_S\ ] \tag{7}$$

where H contains k signal direction vectors. If these directions are all correct, all of the signal components will be nulled out, and only noise will be left. If only k_1 directions are correct, the matrix will have $(k-k_1)$ 'large' eigenvalues and k_1 small eigenvalues (at noise level). In principle, we could search all possible directions and inspect the eigenvalues for all of these directions. To avoid iterations, we would like to have only one direction vector, $\underline{h}$, and perform a single scanned output. This suggests that we find the eigenvalues of the matrix

$$R_r = [(I - \underline{h}\,(\underline{h}^\dagger\underline{h})^{-1}\,\underline{h}^\dagger)R_s] \tag{8}$$

We now note that when $\underline{h}$ is a signal direction there will be (k-1) 'large' eigenvalues. We therefore need to inspect the k'th eigenvalue of Equation (8). When we are in a signal direction, this eigenvalue will decrease to noise level.

This has removed the need for iteration, but the process stillappears to be extremely expensive, because we need to calculate the eigenvalues for all directions. We will now show, however, that this is not as difficult as might be supposed. In order to keep the results general we will calculate the k'th eigenvalue of Equation (6). The eigenvalues are given by the roots μ of the equation.

$$\det[(I - \underline{h}\,(\underline{h}^\dagger\underline{h})^{-1}\,\underline{h}^\dagger)R_s-\mu I] = 0 \tag{9}$$

Using the definition of $\underline{h}$ (Equation (1)), $\underline{h}^\dagger\underline{h} = p$, it is easy to show that this may be rewritten [2]

$$p - \underline{h}^\dagger U_s\Sigma_s(\Sigma_s-\mu I_k)^{-1}U_s{}^\dagger\underline{h} = 0 \tag{10}$$

which can be written

$$p - \sum_{i=1}^{k} \frac{\lambda_i\ |\beta_i|^2}{(\lambda_i-\mu)} = 0 \tag{11}$$

where $\beta_i = \underline{h}^\dagger\underline{u}_i$, and $\underline{u}_i$ is the i'th eigenvector. We are particularly interested in the smallest eigenvalue μ_k which should decrease rapidly (down to the noise level) near to the correct signal directions.

We may multiply out Equation (11) to obtain

$$p \prod_{i=1}^{k} (\lambda_i-\mu_k) - \sum_{i=1}^{k} \lambda_i|\beta_i|^2 \prod_{j\neq i}^{k} (\lambda_j-\mu_k) \approx 0 \tag{12}$$

where Π denotes product. We are interested in the smallest root of this Equation. If we expand in a power series and discard high order terms we find

$$\mu_k \approx \frac{p - \sum_{i=1}^{k} |\beta_i|^2}{p\,W_o - \sum_{i=1}^{k} W_i\ |\beta_i|^2} \tag{13}$$

where

$$W_i = \sum_{\substack{j \neq i \\ j=1}}^{k} 1/\lambda_j \qquad (14)$$

This expression may be compared to MUSIC, which is given by

$$f(\theta) = 1 \Big/ \left[p - \sum_{i=1}^{k} |\beta_i|^2 \right] \qquad (15)$$

There has been a great deal of interest in the weighting of eigenspectra [e.g. 4]. However, Equation (13) is very novel, in that the numerator is identical to MUSIC, but there is also a denominator which has weighted components. It will be seen that this Equation yields very impressive simulation results. However, we may improve the results further by artificially enhancing the signal-to-noise (SNR) ratio. At high SNR, all of the signal eigenvalues become very large. We can therefore enhance the performance by setting the largest eigenvalues to infinity. If we have k_L 'large' eigenvalues, the weightings then become:

$$W_i = \sum_{\substack{j \neq i \\ j=k_L+1}}^{k} 1/\lambda_i \qquad (16)$$

This paper recommends the weightings given in Equation (16) if we require resolution. However, the weightings given in (14) generally lead to more accurate diection estimates, but with reduced resolution performance. However, it is quite possible that different weights might have even better performance. One reason for this is that Equation (16) was derived on the assumption that the signal directions can be detected by changes in the k'th eigenvalue of Equation (9). This clearly has very good performance, but at low SNR we can expected significant leakage between eigenvalues, and a different weighting might be able to restore this information.

3. SIMULATIONS AND DISCUSSION.

In this section we will explore the performance of Equation (13) with the weightings given in (16), under a number of situations. We will concentrate on determining the improvement in the threshold performance, since this is probably the most important property. We will base the simulations on the following example:

(a) 10 element equispaced array (half-wavelength spacing).

(b) 3 signals:

- 80 degrees (0 dB)
- 83 degrees (0 dB)
- 60 degrees (-10 dB)

We will vary the uncorrelated noise power, and examine two exteme cases. First we will use only 5 snapshots, and then we will have a high signal correlation. To provide some statistical stability and yet visualise the results, we plot 10 examples for each case.
In all of the examples, only the largest eigenvalue will be omitted from the weights in Equation (17).

We will use a relatively small number of examples, so that we can plot the individual results. In all cases the results will be so dramatic, that larger numbers of simulations are not required.

Figure 1 explores the performance of Equations (13,16) when there are very few snapshots. Figure 1(a) gives the MUSIC result and 1(b) uses Equations (13,16) for 5 snapshots and a noise power of −23dB. It may be seen that MUSIC usually fails, but Equations (13,16) resolve easily. Figures 1(c) and 1(d) repeat the experiment with −13dB noise. Equations (13,16) are again usually able to resolve, so there is at least 10dB improvement in the threshold.

In many applications, we may expect high signal correlation, and MUSIC is known to fail in these situations. Although the present algorithm cannot resolve perfectly correlated signals, it can resolve them when they have high correlation. To demonstrate this, Figures 2(a) and (b) examines what happens with a correlation of ρ=0.98, with only 5 snapshots, and −37dB of noise. A high SNR is required because of the high signal correlation and the small number of snapshots. MUSIC performs badly, even at this SNR. Figures 2(c) and (d) repeat the experiment at −27dB. Equations (13,16) again shows the 10db improvement in performance, with most cases being resolved. Some of these plots are relatively smooth, but still most contain minima in the correct directions. This smoothness is because the 3'rd eigenvalue of R is <u>very</u> close to the noise eigenvalues, and so very little variation can occur with changes in direction. With fully correlated signals, <u>no</u> variation of the 3'rd eigenvalue can occur, which explains why the method cannot resolve in this case.

These results demonstrate that at least 10dB improvement can be achieved with Equations (13,16). This is a substantial improvement over MUSIC and should make high resolution methods much more useful in applications.

The simulations demonstrate a significant improvement over MUSIC. Because of the similarity to reference [1], good performance with an unknown noise covariance is also obtained. The improvement in performance, combined with the additional robustness, makes it a very useful technique.

Analysis in [2] demonstrates that the new class of algorithms are related to the MUSIC algorithm, and have identical asymptotic performance. This might appear disappointing, but asymptotic performance is not normally of practical importance. The new algorithm can still be expected to out-perform MUSIC in applications.

4. CONCLUSIONS

A method for processing sensor arrays, based on eigenvalue spectra, has been introduced. The method has been shown to be very similar to MUSIC in some respects but to have superior performance to the MUSIC algorithm. The method is effective when there are only a small number of snapshots, or when the signals are highly (but not totally) correlated. The method appears to have a noise threshold which is 10dB lower than for the MUSIC algorithm.

5. REFERENCES

1. J.P.Reilly, K.M. Wong, "Direction of Arrival in the presence of Noise with Unknown Covariance Matrices", ICASSP-90, Glasgow, May 1989.
2. D.R.Farrier, "A High Performance Signal Subspace Beamformer" IEE Proceedings, Part F, accepted and awaiting publication.

Figure 1: Comparison of the MUSIC algorithm ((a),(c)),with Eqns (13,16) ((b),(d)) with 5 snapshots.

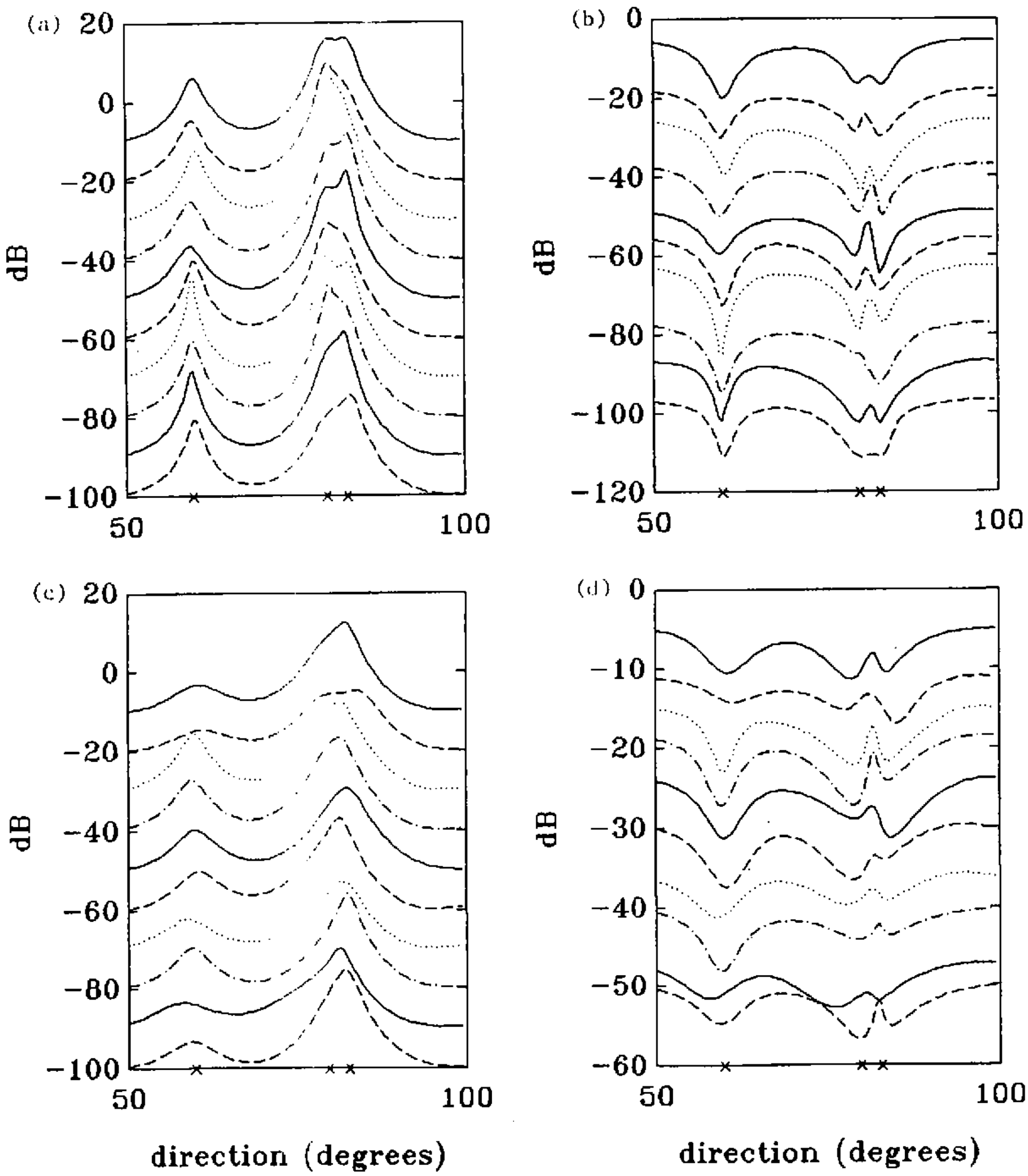

Figure 2: Comparison of the MUSIC algorithm ((a),(c)) with Eqns. (13,16) ((b),(d)) with 0.98 signal correlation