# A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF THE ACOUSTIC FIELD IN THREE-DIMENSIONAL FLUID-STRUCTURE INTERACTION PROBLEMS

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## 1 INTRODUCTION

A fundamental problem in the analysis of sonar transducers is that of modelling the acoustic radiation into the fluid and its associated loading on the transducer surface. The use of fluid finite elements might seem to be a natural extension to a finite-element (FE) treatment of the transducer structure; however, some way must be found of dealing with the effectively infinite extent of the fluid, for example, by ensuring that the conditions at the outer boundary of the fluid elements represent continued radiation [1,2], or by using 'infinite' fluid elements [3]. An alternative approach is to use a boundary-element method (BEM) to model the acoustic radiation, coupled with the finite-element treatment of the transducer structure [4]. There are two immediate advantages: firstly, the BEM inherently treats the fluid as being infinite in extent; and, secondly, the number of dimensions is reduced by 1, since the method solves for the unknowns on the radiating surface rather than throughout the fluid region. In addition, on a practical level, there is no need to devise any further element meshing, as the exposed faces of the structural finite elements provide the required boundary elements.

The acoustic BEM is based on an integral form of the Helmholtz wave equation. Unfortunately this integral equation fails to provide a unique solution at the standing wave frequencies of the region interior to the radiating surface [5]. One practical consequence of this failure is that it can make it difficult to identify genuine resonant frequencies of a submerged transducer. The method chosen here to overcome the problem of non-uniqueness is based on the Helmholtz gradient formulation (HGF), originally proposed by Burton and Miller [6], in which the standard integral equation is combined with its normal derivative form. There are, however, difficulties in the numerical implementation of this formulation. The integrals contain singular kernels; further, if advanced boundary elements are used, the normal derivative form cannot be applied at a node lying at an edge or corner, where the normal direction is not well defined.

In the current work one requirement was that the BEM should be compatible with a finite-element analysis using quadratic isoparametric elements. This paper reports a scheme applicable to such elements, in which the normal derivative form of the Helmholtz integral is applied at the local origin ( or 'centre') of each element rather than at the nodes. The scheme eases the numerical difficulties, and reduces the amount of extra computation inherent in the HGF. Results are presented for radiation from a sphere and from a cylinder, for which analytical or semi-analytical solutions are available. A companion paper [7] reports on the use of this method in a coupled FE-BE analysis of a sonar transducer, and compares the numerical solutions with experimental results.

## 2 THE HELMHOLTZ GRADIENT FORMULATION

The requirement is to solve the Helmholtz wave equation

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

for the acoustic pressure $\phi$ (with time factor $e^{i\omega t}$ understood), where $k = \omega/c$ is the wavenumber, and $c$ is the sound speed in the fluid. This equation can be represented by an integral form, in which the pressure at any point $p$ is obtained in terms of an integral over the radiating surface $S$ [8,9] :

$$\int_S \left\{ \phi(p)\frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q)\frac{\partial \phi(q)}{\partial n_q} \right\} dS_q = \beta(p)\phi(p) \tag{2}$$

A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF ACOUSTIC FIELDS

$G_k$ is the Green's function, which in 3 dimensions is given by

$$G_k(p,q) = \frac{e^{-ikr}}{4\pi r}, \qquad r = |p-q|$$

The value of the factor $\beta$ depends on the position of $p$:

$$\beta(p) = \begin{cases} 0 & p \text{ inside } S \\ 1 & p \text{ outside } S \\ \dfrac{\text{exterior solid angle at } p}{4\pi} & p \text{ on } S \end{cases}$$

For $p$ on $S$, with $S$ smooth at $p$, $\beta = \frac{1}{2}$ ; the more general result above, which allows for sharp edges and corners, is given by Terai [9].

At certain critical frequencies, corresponding to the eigenvalues of the interior Dirichlet problem, the integral equation (2) becomes singular. A second integral equation can be formed by differentiating the first in the normal direction at $p$:

$$\int_S \left\{ \phi(p) \frac{\partial G_k^2(p,q)}{\partial n_p \partial n_q} - \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} \right\} dS_q = \beta(p) \frac{\partial \phi(p)}{\partial n_p} \tag{3}$$

This equation also suffers from singularity, but at a different set of critical frequencies, corresponding to the eigenvalues of the interior Neumann problem. By combining the two equations, a unique solution can be obtained for all frequencies. In the Burton and Miller formulation [6] a multiple $\alpha$ of the second equation is added to the first. To ensure non-singularity in the combined equation, $\alpha$ should have a non-zero imaginary part; a recommended optimum value is $-i/k$ [9].

One of the difficulties of the Burton and Miller formulation is that the first term in the integrand of equation (3) has, for quadratic elements, behaviour of order $1/r^2$ and so is highly singular. The problem is alleviated by using a relationship derived by Meyer et al. [10] :

$$\int_S \phi(q) \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q$$
$$= \int_S [\phi(q) - \phi(p)] \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} dS_q - \phi(p) \int_S (n_p . n_q)(ik)^2 G_k(p,q) dS_q \tag{4}$$

where $n_p$ and $n_q$ are the normals at $p$ and $q$ respectively.

## 3 NUMERICAL IMPLEMENTATION

### 3.1 Boundary element formulation

In the numerical implementation the pressure distribution on S is represented by discrete pressures $\phi_i$ at a set of points $p_i$ ($i = 1$ to $n$) on $S$. By taking the calculation point $p$ at each $p_i$ in turn, $n$ equations are formed relating the $n$ unknown pressures $\phi_i$ to the $n$ unknown pressure gradients $\partial\phi_i/\partial n$, and hence to the normal velocities using the boundary condition

$$\frac{\partial\phi}{\partial n} = -i\omega\rho v_n \tag{5}$$

where $\rho$ is the fluid density.

Because of the need to link the BE analysis to an existing FE program for the structure, quadratic quadrilateral elements were used to discretize the radiating surface, based on the isoparametric FE formulation [11]. Each element is defined by 8 nodes (4 corner, 4 mid-side) and is a mapping from a 2x2 square in $(\xi,\eta)$ space, with the Cartesian coordinates and variables interpolated from the values at the nodes; for example,

$$\phi(\xi,\eta) = \sum_{k=1}^{8} N_k(\xi,\eta)\hat{\phi}_{mk} \tag{6}$$

where $\hat{\phi}_{mk}$ indicates the value of $\phi$ at node $k$ of element $m$. The shape functions $N_k(\xi,\eta)$ are of the standard form for this type of element [4,11]; these produce variables with a quadratic variation over each element.

3.2 Discretized form of the integral equations

With the variables interpolated as above, the standard Helmholtz integral equation can be written in discretized form as

$$-\beta_i\hat{\phi}_i + \sum_{m=1}^{nelem.}\sum_{k=1}^{8} A^i_{mk}\hat{\phi}_{mk} = \sum_{m=1}^{nelem.}\sum_{k=1}^{8} B^i_{mk}.\hat{v}_{mk} \tag{7}$$

for $i = 1$ to $n$, where

$$A^i_{mk} = \int_{S_m} N_k(q)\frac{\partial G_k(p,q)}{\partial n_q}dS_q$$

$$B^i_{mk} = -i\omega\rho\int_{S_m} N_k(q)G_k(p,q)n_q dS_q \tag{8}$$

The integral for $B^i_{mk}$, incorporating the normal vector $n_q$, is given in this form so that the 3-dimensional velocities $\hat{v}_i$ at the nodes can be used rather than the normal velocities, which will not be well defined at sharp edges or corners.

The coefficients $A^i_{mk}$ and $B^i_{mk}$ are assembled into global matrices $A$ and $B$ by summing the coefficients that correspond to the same global node. With the term $-\beta_i$ included on the diagonal of $A$, we can represent the standard Helmholtz integral equation as

$$[A]\{\hat{\phi}\} = [B]\{\hat{v}\} \tag{9}$$

The same process is followed in discretizing the normal derivative form (3) of the Helmholtz equation, except in one major respect. The normal derivative may be discontinuous across element boundaries, and as a consequence the nodes are not suitable as calculation points for the normal derivative equation. The solution proposed in this paper is to apply the normal derivative form at the local origin $(\xi,\eta) = (0,0)$ or 'centre' of each element. For a typical BE model using quadratic isoparametric elements this provides approximately $n/3$ calculation points. The use of a smaller number of points can be justified on the basis of work by Harris and Amini [12], who report that the coupling parameter $\alpha$ in the Burton and Miller formulation may be set to zero over a large part of the surface without adversely affecting the conditioning of the equation. We shall use the term 'partial Helmholtz gradient formulation' to refer to such an approach.

Taking $p$ at the centre of element $i$, and using the integral relationship (4), the normal derivative equation can be written as

$$\sum_{\substack{m=1\\m\neq i}}^{nelem.}\sum_{k=1}^{8}(C_{mk}^{i(q)}\hat{\phi}_{mk}-C_{mk}^{i(p)}\hat{\phi}_{ik}) + \sum_{k=1}^{8}C_{ik}^{(qp)}\hat{\phi}_{ik} + \sum_{m=1}^{nelem.}\sum_{k=1}^{8}C_{mk}^{i(s)}\hat{\phi}_{ik}$$

$$= \sum_{m=1}^{nelem.}\sum_{k=1}^{8}D_{mk}^{i}\hat{v}_{mk} + \frac{1}{2}\sum_{k=1}^{8}n_pN_k^0\hat{v}_{ik} \qquad (10)$$

where

$$C_{mk}^{i(q)} = \int_{S_m}N_k\frac{\partial G_k(p,q)}{\partial n_p\partial n_q}dS_q \quad (m\neq i), \qquad C_{mk}^{i(p)} = N_k^0\int_{S_m}\frac{\partial G_k(p,q)}{\partial n_p\partial n_q}dS_q \quad (m\neq i),$$

$$C_{mk}^{i(qp)} = \int_{S_m}(N_k-N_k^0)\frac{\partial G_k(p,q)}{\partial n_p\partial n_q}dS_q, \qquad C_{mk}^{(s)} = k^2N_k^0\int_{S_m}G_k(p,q)n_p.n_qdS_q,$$

$$D_{mk}^{i} = -i\omega\rho\int_{S_m}N_k\frac{\partial G_k(p,q)}{\partial n_p}n_pdS_q, \qquad N_k^0 = N_k(0,0) \qquad (11)$$

Assembly of the coefficients into global matrices leads to an equation of the form

$$[C]\{\hat{\phi}\} = [D]\{\hat{v}\} \qquad (12)$$

3.3 Evaluation of the integrals

The integrals in equations (8) and (11) are evaluated by transforming to the $(\xi,\eta)$ plane, so that for example,

$$A_{mk}^{i} = \int_{\eta=-1}^{1}\int_{\xi=-1}^{1}N_k(\xi,\eta)\frac{\partial G_k(p,q)}{\partial n_q}J(\xi,\eta)d\xi d\eta \qquad (13)$$

where $J(\xi,\eta)$ is the Jacobian of the transformation from Cartesian to $(\xi,\eta)$ coordinates. Provided the kernels are non-singular, numerical evaluation is carried out using Gauss quadrature. For the integrals with singular kernels, namely $A_{ik}^{i}$ and $C_{ik}^{(pq)}$, the square in $(\xi,\eta)$ space representing the singular element is subdivided into triangles, using the calculation point $p$ as a common vertex, following a technique of Lachat and Watson [13]. For each triangle, transformation to a local polar coordinate system with the origin at $p$ produces a Jacobian with $O(r)$ behaviour, so removing the $O(1/r)$ singularity in the kernel of $A_{ik}^{i}$. For the kernel of $C_{ik}^{(pq)}$, the factor $N_k-N_k^0$ is either zero or has $O(r)$ behaviour, so that the $O(1/r^2)$ singularity in $\partial^2G_k/\partial n_p\partial n_q$ is again removed. Evaluation of the integrals can then be carried out using Gauss quadrature in the triangular coordinate system.

3.4 Combining the integral equations

The system of equations (9) derived from the standard Helmholtz integral equation is based on taking calculation points at the nodes $p_i$, whereas the system (12) derived from the normal derivative equation uses calculation points at the centres of the elements. The two systems cannot therefore be combined directly as in the original Burton and Miller formulation. The method of combination adopted here is based on the association of each normal derivative equation with the 8 nodes of the element on which the calculation point for that equation is located. This association is preserved by adding $\alpha$ times the normal derivative equation to each of the 8 equations in the standard form which have their calculation points at those 8 nodes. From the point of view of the standard system, this means that the $i$ th row has added to it $\alpha$ times each of the normal derivative equations corresponding to the elements which meet at node $i$. The value of $\alpha$ has been taken to be $-i/km_i$ where $m_i$ = number of elements meeting at node $i$.

## 4 RESULTS

The first test case considered is that of an oscillating sphere, i.e., a sphere acting as a dipole source, for which an analytical solution is available. Two boundary-element models were used, a coarse one of 24 elements, and a refined one of 96 elements. Figures 1 and 2 show plots of the normalized surface pressure amplitude at the pole of the sphere (on the axis of oscillation) plotted against the dimensionless wavenumber $ka$, where $a$ = radius of sphere. One of the critical frequencies for this problem occurs when $ka$ = 4.493..., and the range for $ka$ has been taken around this value. Results are shown for both the standard Helmholtz formulation, and for the partial HGF as described above; the theoretical solution is included for comparison.

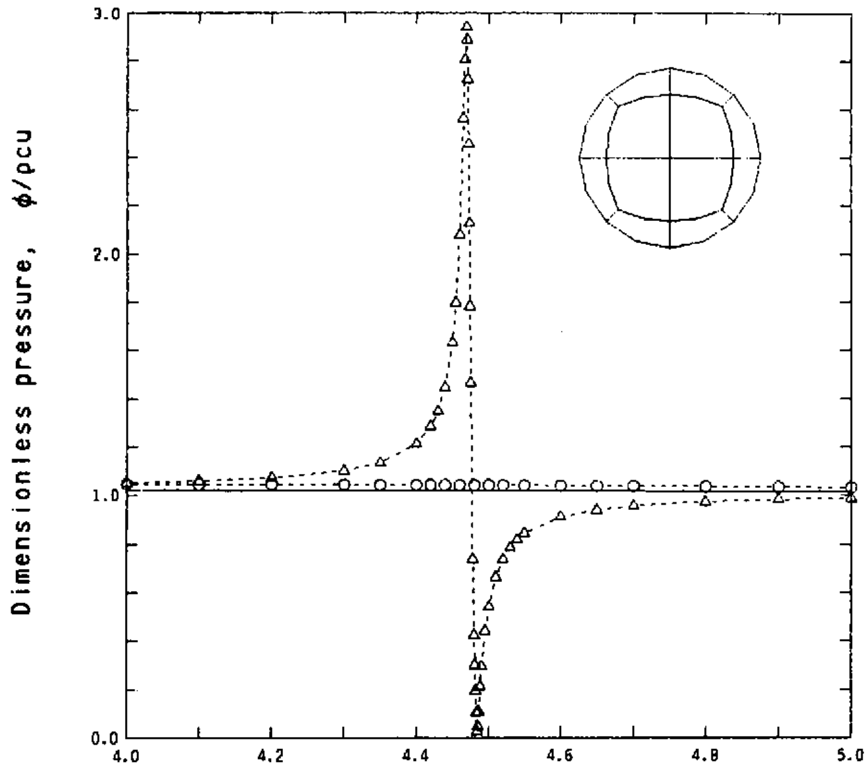


Figure 1. Amplitude of the acoustic pressure at the pole of an oscillating sphere vs. $ka$, 24-element model : —— analytical solution; △ standard Helmholtz formulation; o partial Helmholtz gradient formulation.

A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF ACOUSTIC FIELDS

The effect of ill-conditioning in the standard formulation can be seen in the irregularity of the solutions around the critical frequency. The partial HGF, on the other hand, gives consistent results across the critical frequency range.

The results illustrate a point noted elsewhere [14], that with better numerical techniques, such as the use of advanced elements, the failure in the standard formulation occurs in a narrow band. Comparison of Figures 1 and 2 (note the expanded scale in Figure 2) reveals that the narrowness of the band depends on the fineness of the model. However, while the use of a finer model reduces the risk of ill-conditioning, it does not eliminate that risk, and it requires more computing time than the partial HGF applied to the coarser model.

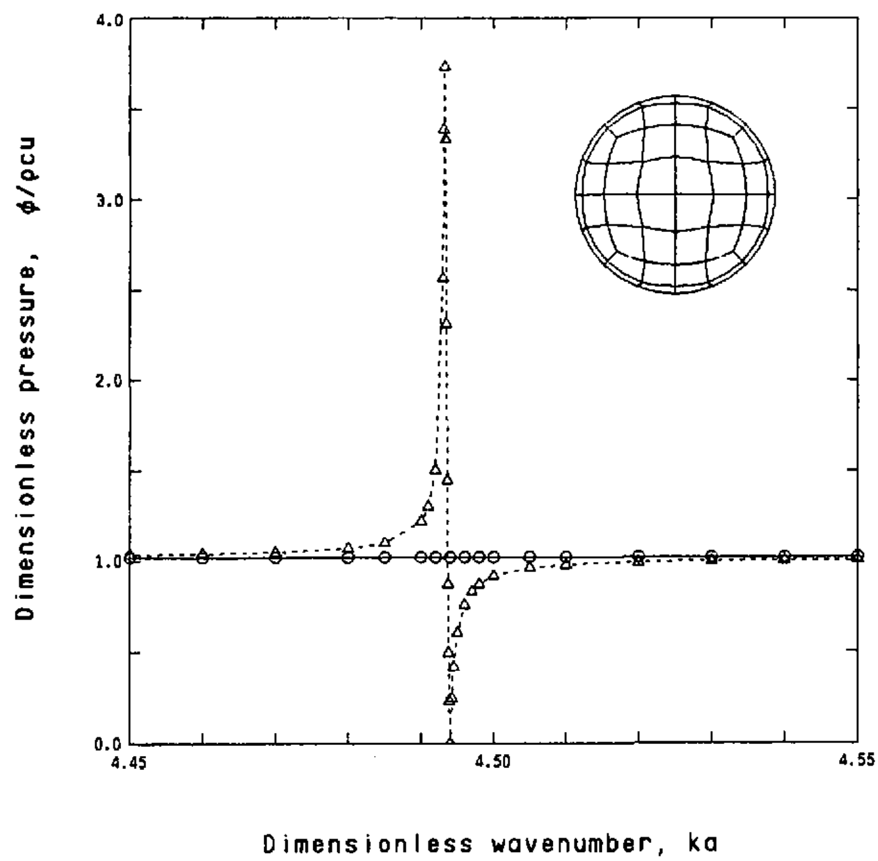


Figure 2. As Figure 1, but for 96-element model.

A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF ACOUSTIC FIELDS

Testing the capability of the BE method for non-smooth surfaces presents some difficulty because of the absence of analytical solutions. It is possible, however, to obtain a semi-analytical solution, in the form of a series of eigenfunctions, for a cylinder with a prescribed velocity distribution on the surface [15]. The solution varies in its accuracy depending on how well the velocity distribution can be represented in the series form. The problem considered here is that of a cylinder of radius $a$ and length $2a$, with pistons of radius $a/2$ concentrically placed on the end faces; the pistons are oscillating in opposite directions to each other with uniform velocity amplitude $u$, the rest of the surface being rigid. Figure 3 compares the results obtained using the partial HGF with the series solution; the sound pressures at a distance of $10a$ from the centroid of the cylinder are plotted against the angular position, as measured from the axis of the cylinder. Because of the symmetry of the problem, the results are shown for just one quadrant. The two solutions agree well over most of the quadrant; the divergence near the axis can be explained by the tendency of the series solution to produce higher-than-specified velocities near the centres of the pistons. It should be noted that in the BE model, the element dimension (typically $0.3a$) is close to half the wavelength ($\pi a/5$ for $ka = 10$), beyond which the quadratic formulation ceases to be a good approximation.

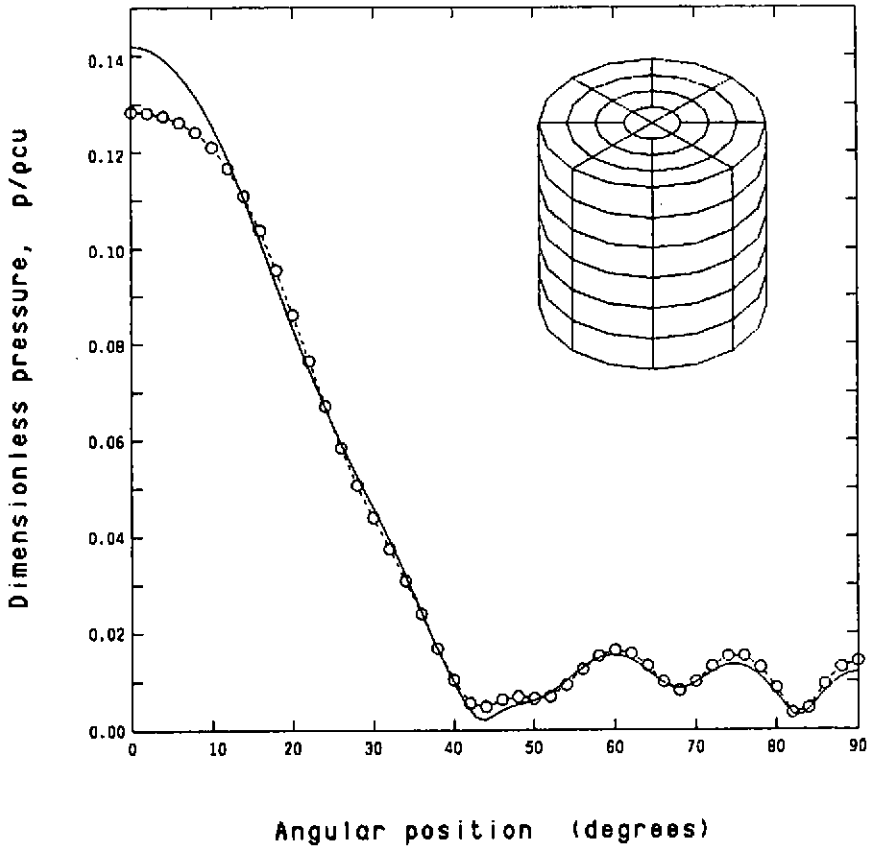


Figure 3. Sound pressure amplitude vs. angular position at a distance of $10a$ from the centroid of a cylinder of radius $a$, length $2a$, with normal velocity $u$ on pistons of radius $a/2$ concentrically placed on the end faces; $ka = 10$; 104-element model: —— series solution ( [15], $N = 40$ ); o BEM, partial HGF.

A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF ACOUSTIC FIELDS

Figure 4 illustrates how the partial HGF performs over a range of frequency, in particular how it overcomes the singularity problem in the standard formulation. The radiation impedance, defined [16] by

$$Z_s = \frac{\int_S \phi v_n^* dS}{\int_S |v_n|^2 dS}$$

has been chosen as it provides on overall measure of the acoustic behaviour of the surface; it has been calculated for a uniformly vibrating piston of radius $a/2$ set concentrically on the end face of a cylinder of radius $a$ and length $2a$, i.e., the configuration is the same as in the previous problem except that only one end of the cylinder has an active section. The real and imaginary parts of the impedance values obtained from the standard formulation are shown as individual points, while for clarity those from the partial HGF are shown by continuous lines based on discrete values at intervals of 0.1 in $ka$. The irregularities in the results for the standard formulation are not so dramatic as in the case of the sphere (Figures 1 and 2); however, they would be a considerable nuisance in a combined FE-BE analysis concerned with finding the resonances of a submerged structure. There is no sign of any irregularity in the results obtained using the partial HGF.

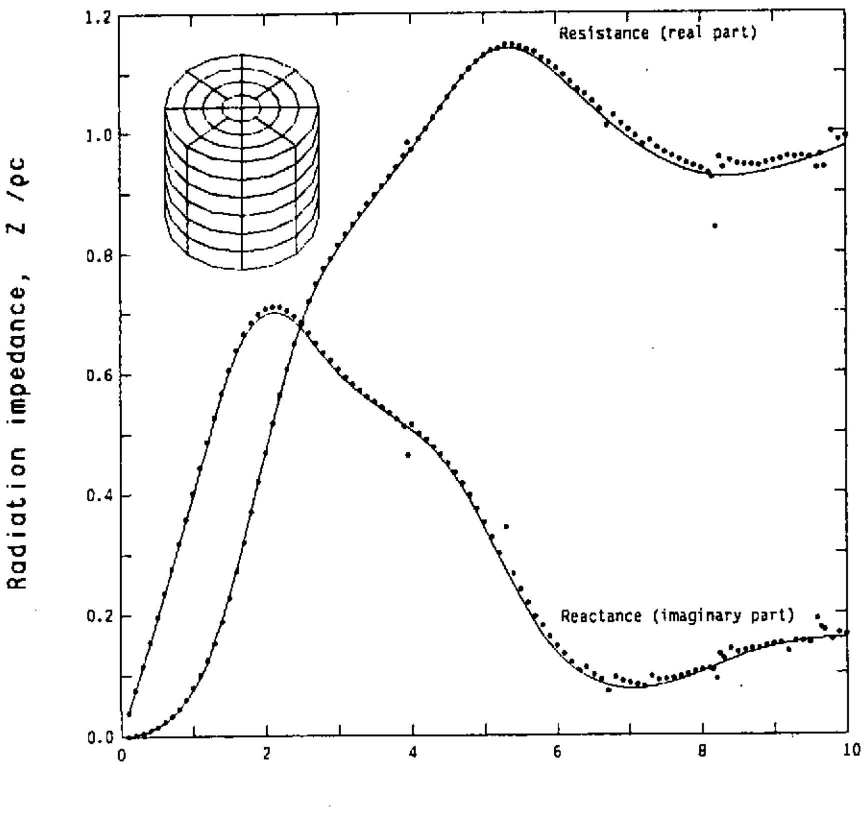


Figure 4. Radiation impedance as a function of frequency for a piston of radius $a/2$ in the end of a cylinder of radius $a$, length $2a$ : ••• standard Helmholtz formulation ; —— partial HGF.

A BOUNDARY-ELEMENT METHOD FOR THE ANALYSIS OF ACOUSTIC FIELDS

## 5 CONCLUSIONS

The standard formulation of the Helmholtz integral equation suffers from singularity at certain frequencies, and this causes irregularities in the numerical solution. The Burton and Miller formulation, where the Helmholtz integral equation is combined with its normal derivative, provides a means of overcoming this problem, but is difficult to apply in its original form if advanced boundary elements are used. Here we have shown that applying the normal derivative equation at the centres of elements rather than at the nodes is sufficient to render the combined equation well-conditioned. This scheme, referred to as a partial Helmholtz gradient formulation, has been implemented numerically for surfaces modelled by quadratic isoparametric elements, and is therefore suitable for combining with a FE analysis where such elements are used [7].

## 6 ACKNOWLEDGEMENT

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