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SCATTERING OF WAVES BY REFRACTIVE LAYERS WITH POWER LAW SPECTRA

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1. INTRODUCTION

Many useful insights into the phenomenology of wave propagation through extended inhomogeneous media have been obtained by studying the properties of a much simpler scattering system: the random phase changing screen. This system is of interest in its own right as a physical optics model for scattering by thin diffusing layers and rough surfaces both in transmissive and reflective geometries. In recent years, laboratory measurements of the scintillation of laser light scattered from turbulent plumes, mixing layers, mobile and rigid rough surfaces [1] have allowed more quantitative assessment of the predictions of a variety of theoretical phase screen models. As a result there has been renewed interest in the physical meaning of various statistical and spectral models and the mathematical implications of using them in phase screen calculations or indeed in the more complicated extended medium problem.

If, as is usually assumed, the phase distortions introduced by the screen constitute a Gaussian Process, then interest centres on the choice of phase autocorrelation function or spectrum. It is well known that autocorrelation functions which can be expanded in an even powered series about the origin, such as Gaussian or Lorentzian models, correspond to smoothly varying single scale phase functions which are infinitely differentiable. In the case of strong scattering, when the path fluctuations exceed a wavelength, non-Gaussian intensity patterns generated by screens of this type are dominated by geometrical optics effects [2]. On the other hand it is now recognised that raw power law models (ie without inner and outer scales) constitute the simplest class of multiscale screens. In this case the phase function is hierarchical, being self-affine under magnification and can be described in the language of Mandelbrot as a Gaussian random fractal [3]. Within this group of models further classification according to spectral index is necessary to distinguish between continuous functions which are not differentiable and those which are once, twice ... or n times differentiable. Clearly the physical implication of a model which is not differentiable will be the absence of geometrical optics effects: the predicted statistical properties of a scattered wave will include only the effects of diffraction and interference [4]. On the other hand a model which is only once differentiable will generate density fluctuations of geometrical rays but no caustics or focusing [5]. Evidently the spectral index and hence the truncation of differentiability determines the maximum order of singularity or catastrophe in the scattered wave field. In practice raw power law behaviour is not observed in nature: often regions of different power law index are found together with high and low frequency cut-offs (inner and outer scale effects). The presence of an outer scale always ensures that when sufficient area of the scatterer contributes to the wave field at the detector then Gaussian field statistics will be observed, the exact nature of the approach to this limit being determined by detail of the low frequency cut-off. The presence of an inner scale means that at sufficient magnification (small enough wavelength) the phase function will appear to be a smoothly varying object. Note, however, that the effect of a high frequency cut-off may often be masked at long wavelengths so that characteristics of the power law region will be present in the scattered wave field.

It has been demonstrated in recent years that amongst the corrugated power law models there exist special cases which can be exactly solved in certain scattering configurations [6]. These are the Brownian fractals generated by simple random walks along a line. Both Brownian fractal phase and slope models have been investigated in some detail [4,6]. Relationships have been established with telegraph wave phase screens, which are non-Gaussian [7], and thence with certain mathematically related problems in statistical mechanics [8]. In fact these exactly solvable models are members of a class of objects in which increments of phase, slope etc are independent. By exploring such models, new insight into the phenomenology of wave propagation may be obtained.

In this paper we present some further results on the statistical properties of ray density fluctuations beyond a refractive layer. In the next section we investigate the effect of spatial integration at the detector when the slope of the scattering layer is fractal. In section 3 additional results on scattering models with independent increments will be derived with a summary and conclusions in the final section.

2. INTEGRATED RAY DENSITY FLUCTUATIONS

The density of rays beyond a diffusing layer which introduces wavefront distortions of local slope $m(x)$ is given by

$$R(y,z) = \frac{1}{z}\int_{-\infty}^{\infty} dx\ \delta\left(m(x) - \frac{x-y}{z}\right) \qquad (1)$$

where z is the propagation distance and y is the lateral displacement coordinate of the detector. We have assumed for simplicity that the phase screen is corrugated and infinite and note that $\langle R\rangle = 1$. In what follows we shall adopt a stationary multivariate Gaussian model for the statistics of m with a power law structure function of the form

$$S(x) = \langle (m(0)-m(x))^2\rangle = |x/L|^\nu \qquad 0 < \nu < 2 \qquad (2)$$

where L is a length scale. It is not difficult to show from equation (1) that the ray density autocorrelation function is given by

$$\langle R(0)R(y)\rangle = \frac{1}{z\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dx}{S(x)}\exp\left\{-\frac{(x+y)^2}{2z^2S(x)}\right\} \qquad (3)$$

where we have suppressed the z coordinate on the left hand side for brevity. This formula can only be evaluated exactly for the model (2) when $\nu = 1$. In this case it has been shown [6] that R is the square modulus of a complex Gaussian-Markov process of decay length z^2/L, for which the integrated statistics are well known. Defining the integrated ray density by

$$E(y;L) = \frac{1}{W} \int_{y-W/2}^{y+W/2} R(y')\, dy' \quad , \tag{4}$$

which corresponds to a slit detector of length W, we find the generating function for the distribution of E is given by [9]

$$Q(s) = \langle exp(-sE) \rangle = (exp\,\gamma)\left[cosh\ \eta + \frac{1}{2}\left(\frac{\gamma}{\eta} + \frac{\eta}{\gamma}\right) sinh\ \eta\right]^{-1} \tag{5}$$

where $\gamma = LW/z^2$

and $\eta^2 = \gamma^2 + 2\gamma s$

Moments of the distribution can easily be obtained from this formula by evaluating derivatives with respect to s at $s = 0$. For example

$$\langle E^2 \rangle = 1 + (2\gamma)^{-1}\ [2 - \gamma^{-1} + \gamma^{-1}\ exp(-2\gamma)] \tag{6}$$

The generating function for the joint distribution of integrated ray densities is also known so that the effect of detector integration on correlation properties can be calculated. The simplest result for the bivariate moment is given for $|y-y'| > W$ by [10]

$$\langle E(y)E(y') \rangle = 1 + \frac{sinh^2\ \gamma}{\gamma^2}\ exp(-2L|y-y'|/z^2) \tag{7}$$

As ν approaches 2 the integral on the right hand side of equation (3) diverges, reflecting the onset of caustics or singularities in the ray density pattern in this limit. We report an investigation of the nature of the divergence elsewhere. An alternative approach is to set $\nu = 2$ before taking the short wave limit and examine the divergence of the wave amplitude statistics with wave vector [2]. We explore yet another avenue here by observing that the smoothing provided by detector integration should ameliorate the effect of singularities. In fact we find finite results and make contact with previous work on smoothly varying scattering models only when a finite outer scale is included in the model (2) in addition to detector integration. Instead of the "hard" aperture (4) a Gaussian-profile "soft" aperture is employed for mathematical simplicity and we calculate the simplest integrated statistic

$$\langle E^2 \rangle = \frac{1}{2W\sqrt{\pi}} \int_{-\infty}^{\infty} dy\ exp(-y^2/4W^2)\ \langle R(0)\ R(y) \rangle \tag{8}$$

where $\langle R(0)R(y)\rangle$ is given by (3) and

$$S(x) = x^2/L^2 \qquad |x| < \xi$$
$$= \xi^2/L^2 \qquad |x| \geqslant \xi \tag{9}$$

This leads to the exact result

$$\langle E^2\rangle = 1 - erf(q) + \frac{q}{\sqrt{\pi}}\, exp(-q^2)\left[ln\left(\frac{2z^2\xi^2}{L^2W^2}\right) + \pi\ erfi(q) - F(q)\right] \tag{10}$$

where $F(q) = \int_0^q exp(x^2)\ erfc(x)\ dx$

and $\qquad q = L/z\sqrt{2}$

This formula bears a remarkable resemblance to an approximate expression obtained previously from finite wavevector calculations of the second moment of intensity fluctuations in waves scattered by a smoothly varying phase screen. In fact, with finite wave vector (k), the second moment of intensity fluctuations beyond a Gaussian phase screen with Gaussian correlation function is [11]:

$$\langle I^2\rangle = 2(1-erf\bar{q}) + \frac{\bar{q}}{\sqrt{\pi}}\, exp(-\bar{q}^2)\left\{ln\left(\frac{8\gamma k^2h_o^2}{3}\right) + \pi erfi(\bar{q}) - F(\bar{q})\right\} \tag{11}$$

where $\bar{q} = \bar{\xi}^2/2zh_o\sqrt{6}$, kh_o is the rms phase fluctuation, $\bar{\xi}$ the characteristic correlation length and γ is the Euler constant. Equations (10) and (11) are structurally identical except for the factor of 2 appearing before the first term on the right hand side of equation (11). This factor is associated with speckle or interference and we would not expect it to appear in expressions for ray statistics. Comparison of the expansion of a Gaussian correlation function about the origin with model (9) shows that $\xi = \bar{\xi}$ and $L^2 = \xi^4/12h_o^2$ confirming that the geometrical parameters q and $\bar{q}$ are the same. Inspection of the logarithmic terms then reveals that, apart from constants, these are identical if the detector size W is equivalent to the diffraction scale $z/k\xi$. Thus, integration at the detector is playing the same role as diffraction smoothing in these calculations.

3. SCREENS WITH INDEPENDENT INCREMENTS

In this section we continue the theme of ray density fluctuations, concentrating on the Brownian fractal slope model (2) with $\nu = 1$. A simplification of the analysis presented in previous work, which enables new results to be derived, is achieved by taking advantage of the Markovian factorisation of the joint slope distribution implicit in this model:

$$p(m_1,m_2,m_3\ \dots) = p_o(m_1)\ p(m_2-m_1)\ p(m_3-m_2) - \dots \tag{12}$$

where $m_j = m(x_j)$ with $x_j > x_{j-1}$ and $p_o(m_1)$ strictly speaking has infinite width. Note that the distributions p and p_o are Gaussian. If equation (1) is generalised to include a hard aperture centred at the origin

$$R(y) = \frac{1}{z}\int_{-W/2}^{W/2} dx\ \delta\left(m(x) - \frac{x-y}{z}\right)$$

then

$$\langle R^N\rangle = \frac{1}{z^N}\int_{-W/2}^{W/2}\!\!\cdots\!\int dx_1 \cdots dx_N\ p\left(m_1 = \frac{x_1-y}{z},\ m_2 = \frac{x_2-y}{z} \cdots m_N = \frac{m_N-y}{z}\right) \tag{13}$$

The region of integration can be divided into $N!$ volumes giving equal contributions where the $\{x_j\}$ are differently ordered. Using the factorisation property (12) we obtain, after some manipulation,

$$\langle R^N\rangle = \frac{z}{L}N!\int_0^{LW/z^2} dx_1\ p_o\left(\frac{W}{2z} - \frac{zx_1}{L} - \frac{y}{z}\right)\int_0^{x_1} dx_2\ \frac{p(\sqrt{(x_1-x_2)})}{\sqrt{(x_1-x_2)}}\int_0^{x_2}\frac{p(\sqrt{(x_2-x_3)})}{\sqrt{(x_2-x_3)}}\cdots$$

$$\cdots\int_0^{x_{N-1}} dx_N\ \frac{p(\sqrt{(x_{N-1}-x_N)})}{\sqrt{(x_{N-1}-x_N)}} \tag{14}$$

where $p(x) = exp(-x^2/2)/\sqrt{2\pi}$ and $p_o(x) = exp(-x^2/2m_o^2)/\sqrt{2\pi m_o^2}$ (15)

$m_o^2 = \xi/2L$ being the slope variance, limited only by the asymptotically large outer scale ξ. The convolution on the right hand side of (14) can be reduced with the help of Laplace transform theory to give the results

$$\langle R^N\rangle = N!\,\frac{z}{L}\int_0^{LW/z^2} dx\ p_o\left(\frac{W}{2z} - \frac{zx}{L} - \frac{y}{z}\right)\mathcal{L}_x^{-1}\left\{\frac{g^{N-1}(p)}{p}\right\} \tag{16}$$

$$P(R) = \delta(R)\left[1 - \frac{z}{L}\int_0^{LW/z^2} dx\; p_o\left(\frac{W}{2z} - \frac{zx}{L} - \frac{y}{z}\right)\mathcal{L}_x^{-1}\left\{\frac{1}{pg(p)}\right\}\right]$$

$$+ \frac{z}{L}\int_0^{LW/z^2} dx\; p_o\left(\frac{W}{2z} - \frac{zx}{L} - \frac{y}{z}\right)\mathcal{L}_x^{-1}\left\{\frac{exp(-R/g(p))}{p^2 g(p)}\right\} \qquad (17)$$

where $\mathcal{L}_x^{-1}\{f(p)\}$ is the inverse Laplace transform $\int_{c-i\infty}^{c+i\infty} exp(px)\, f(p)\, dp$ and $g(p) = 1/\sqrt{2p+1}$. The transform in equation (16) is an incomplete γ-function and those in equation (17) can be expressed in terms of error functions. When $LW/z^2 \gg 1$ the main contribution to the integrals in (17) comes from the region where x is large and we recover the previous results [6] $P(R) \rightarrow exp(-R)$ and $\langle R^N\rangle \rightarrow N!$. However, the general results (16) and (17) allow other regimes to be explored. In particular in the narrow beam limit, $LW/z^2 \ll 1$, new simple results for the distribution of ray density and moments can be obtained:

$$p(R) = (1-2\beta/\sqrt{\pi})\;\delta(R) + (\beta/\alpha)\; erfc(R/2\alpha) \qquad (18)$$

$$\frac{\langle R^N\rangle}{\langle R\rangle^N} = \frac{N!}{\Gamma\left(\frac{N+3}{2}\right)}\beta^{(1-N)}; \qquad \langle R\rangle = \alpha\beta \qquad (19)$$

where $\alpha = \sqrt{LW/2z^2}$, $\beta = \sqrt{2W/L}\; p_o(x/z)$, with p_o given by equation (15). A numerical evaluation of the normalised second moment on axis (y = 0) given by equation (16) is plotted against α^{-1} for various values of β^{-1} in figure 1. This shows how the ray density fluctuations increase with distance from the scatterer and eventually saturate as predicted by result (19).

The foregoing analysis demonstrates the relative simplicity of the Brownian fractal slope model. The new narrow beam results (18) and (19) are entirely consistent with previous conjectures concerning the nature of the scattering in this configuration: the incident beam of rays is steered and broadened by the screen so that, as indicated by the delta-function term in equation (18), for some fraction of the time no energy falls on the detector. The dependence of the statistics on the single interval slope distribution p_o has often been predicted from geometrical considerations and seems to be supported by experimental evidence [12]. The detailed N-dependence of the moments has not been predicted before, however, and warrants comparison with data on fluid scattering systems generating highly non-Gaussian fluctuations.

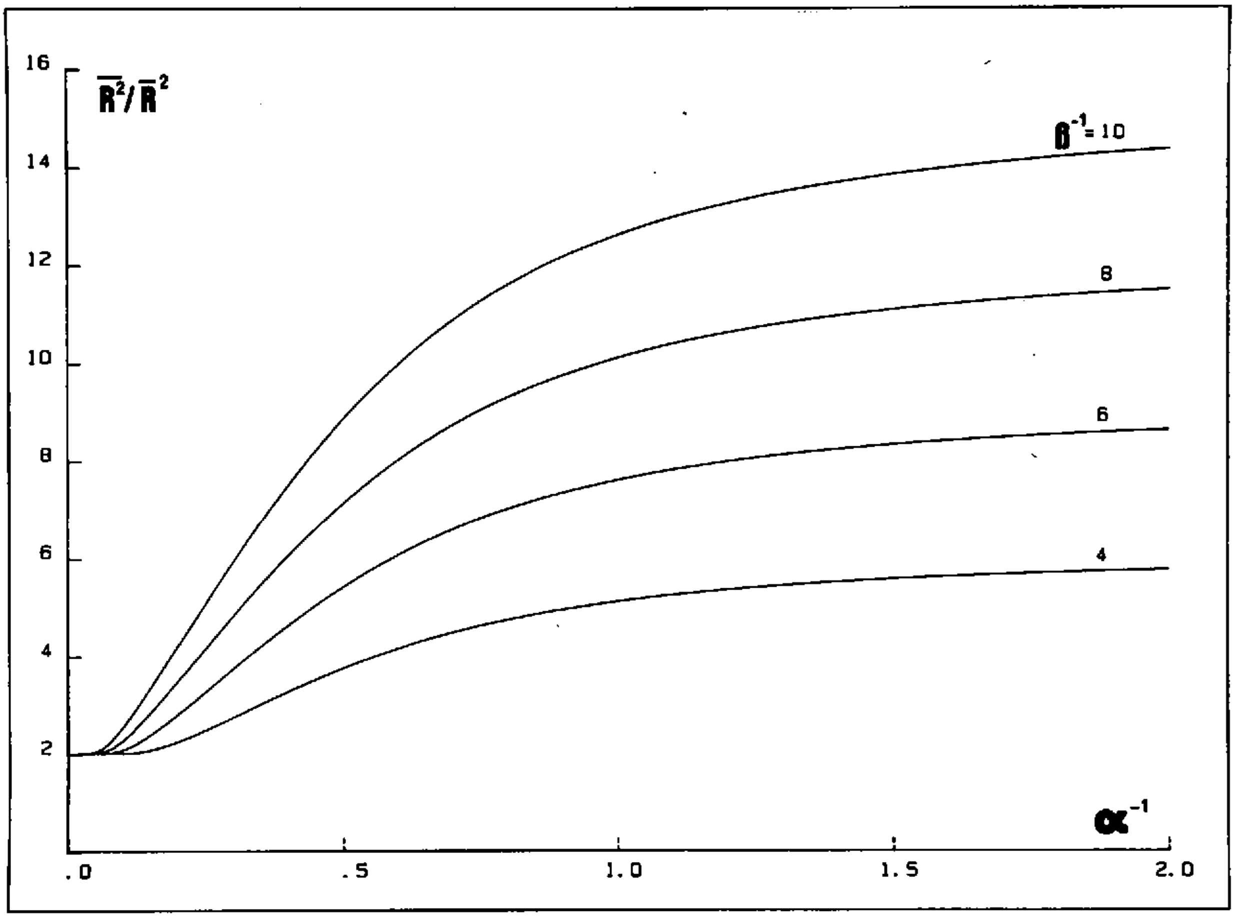

Figure 1: Second normalised moment of ray density fluctuations on axis plotted as a function of scaled distance for various values of a scaled aperture parameter.

5. SUMMARY AND CONCLUSIONS

We have investigated the effect of spatial integration on ray density fluctuations beyond a diffuser with fractal slope and shown how this can be used to examine the marginal case or smoothly varying limit when caustics are generated in the ray density pattern. We have also explored more specifically the Brownian fractal slope model and shown that the equivalent assumption of independent slope increments simplifies the analysis and allows new results to be derived. It is likely that such an assumption for the phase increments would also lead to some simplification in the analogous fractal phase scattering problem but this requires further examination. Progression to models where the phase is not a Gaussian process can be made through generalisations of the kind of random walk construction for the scattering object that we have adopted here [7].

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