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ACOUSTIC WAVE PROPAGATION IN RANDOM MEDIA USING THE PARABOLIC EQUATION AND GREEN'S FUNCTION

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INTRODUCTION

Interest in studies of wave propagation in random media has increased in recent years in connection with their numerous applications in acoustics, biophysics, optics, radiophysics, atmospheric physics, etc [1,2]. The problems associated with random media may be grouped into three main categories: discrete random scatterers, random continua, and rough surfaces [3,4]. Scattering of the waves by the inhomogeneities of the medium or surface, causes stochastic changes in the amplitude and phase of the waves, and - in the case of transverse waves - of the direction of polarization. A variety of mathematical methods have been employed in studies of wave propagation in random media. The choice of the one which is actually used depends to a large extent on the scale of the inhomogeneities in relation to the wavelength of the radiation, and whether multiple scattering needs to be considered at some stage in the analysis.

The present paper reviews briefly the most common methods that are used, namely the methods of optical geometry, and small and smooth peturbations (the last two are sometimes called the Born and Rytov approximations). There are two distinct (but related) results for the analyses. One concerns the average (square) fluctuations that may be observed in the amplitude or phase of the waves that are received, while the other concerns the average loss of amplitude due to scattering by the inhomogeneities. These two types of results are closely related to the concepts of incoherent and coherent scattering that have been used by a number of authors.

The results obtained by the methods given above are compared with those obtained using two different methods - that of the parabolic equation and that using Green's functions. Before summarizing the established results, some definitions need revision.

STATISTICAL PROPERTIES OF A RANDOM MEDIA

The main quantity used in these discussions is the mean square fluctuation in the acoustic refractive index, and its autocorrelation. The square of the refractive index is separated into the mean and fluctuating parts:

$$n^2(\underline{r}) = 1 + \epsilon(\underline{r}) \tag{1}$$

and the autocorrelation of the fluctuations is given by

$$\Psi(\underline{r}_1,\underline{r}_2) = \langle\epsilon(\underline{r}_1)\epsilon(\underline{r}_2)\rangle \tag{2}$$

where the brackets <> indicate an ensemble average. The most common forms for the corelation function are

$$\text{Gaussian : } \Psi(\underline{r}_1,\underline{r}_2) = \langle\epsilon^2\rangle \exp - (r_1-r_2)^2/a^2 \quad (3a)$$

$$\text{or exponential : } \Psi(\underline{r}_1,\underline{r}_2) = \langle\epsilon^2\rangle \exp - |\underline{r}_1-\underline{r}_2|/a \quad (3b)$$

where a is the radius of correlation.

Sometimes the separation of n is $n = 1+\mu$, when $\epsilon=2\mu$ to first order terms.

The correlation coefficient of the medium refractive index variations is given by

$$N = \frac{\langle\mu(\underline{r}_1)\ \mu(\underline{r}_2)\rangle}{\langle\mu^2\rangle} \quad (4)$$

SUMMARY OF ESTABLISHED RESULTS

The method of optical geometry (ray theory)
This is used when the mean size of the inhomogeneities, a, is much greater than the wavelength and if $\lambda L \ll a^2$ where L is the observation distance (which must thus be small compared to a). The mean square phase fluctuations are given by:

$$\langle(\Delta\phi)^2\rangle = \sqrt{\pi}\ \langle\mu^2\rangle\ k^2\ La \quad (5a)$$

$$\text{and } \langle(\Delta\phi)^2\rangle = 2\langle\mu^2\rangle\ k^2\ La \quad (5b)$$

for Gaussian and exponential refractive index correlation coefficients respectively.

The method of smooth perturbations
Rytov's method requires less restricting conditions than the method of small perturbations described in the next subsection in that it requires that the relative changes of amplitude and phase be small only over the scale of a wavelength. The scattered wave (for an incident plane wave) is written in the form:

$$p = A(\underline{r})\ \exp i\ (\omega t - S(\underline{r}))$$

where $A(\underline{r})$ and $S(\underline{r})$ incorporate the amplitude and phase fluctuations. A new function $\Psi(\underline{r})$ is defined such that

$$p = A_o\ \exp i(\omega t - \Psi(\underline{r})) \quad (6)$$

$$\text{where } \Psi(\underline{r}) = S(\underline{r}) - i\ \ln\left(\frac{A(\underline{r})}{A_o}\right)$$

Inserting the solution (6) into the wave equation, taking the first approximation and using the restriction mentioned in the first sentence one finds that the results depend on the parameter $D = 4L/ka^2$.

For D<<1 the results of the ray theory given in the previous section (equations 5a and b) are recovered.

For D>>1 one obtains

$$\langle S^2\rangle = \left\langle\left[\ln\left(\frac{A(r)}{A_o}\right)\right]^2\right\rangle = \langle\mu^2\rangle k^2 L\int_0^\infty N(\xi)d\xi \qquad (7)$$

This gives, for Gaussian and exponential correlation functions respectively:

$$\langle S^2\rangle = \tfrac{1}{2}\sqrt{\pi}\ \langle\mu^2\rangle k^2 aL \qquad (8a)$$

and $$\langle S^2\rangle = \langle\mu^2\rangle k^2 aL \qquad (8b)$$

The method of small perturbations

This is limited to the case in which the relative changes in amplitude and phase are small over the whole of the wave path. The starting point is the wave equation in an inhomogeneous medium [5]:

$$\nabla^2 p - \frac{(1+\mu)^2}{Co^2}\frac{\partial^2 p}{\partial t^2} = (\nabla \ln p)\nabla p \qquad (9)$$

If the wavelength is large compared to the mean scale of the inhomogeneity, i.e. ka<<1, the scattering is isotropic (in an isotropically random medium), while if ka>>1, most of the energy is concentrated in the solid angle $\theta \sim (ka)^{-1}$.

One obtains expressions for the mean intensity attenuation coefficient, ∂, for Gaussian and exponential refractive index correlation functions respectively:

$$\gamma = \sqrt{\pi}\ \langle\mu^2\rangle k^2 a\left[1 - e^{-k^2a^2}\right] \qquad (10a)$$

and $$\gamma = \frac{8\langle\mu^2\rangle k^4 a^3}{(1+4k^2a^2)} \qquad (10b)$$

Thus for ka<<1 , $\gamma \sim k^4$
and for ka>>1 , $\gamma \sim k^2$.

THE PARABOLIC EQUATION METHOD

The parabolic equation method is used if the wavelength is small in comparison with a, i.e. ka>>1 [6,7]. Equation (9) reduces, in the short wavelength approximation [8], to:

$$\nabla^2 p + k^2(1+\mu^2)p = 0$$

for a monochromatic wave, where $k = \omega/C_o$.

Assuming $p = A_o \exp ikx$ for x<0 and $p = A(x,y,z) \exp ikx$ for x>0, we obtain

$$\frac{\partial^2 A}{\partial x^2} + 2ik\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} + k^2(2\mu+\mu^2)\,A = 0 \qquad (11)$$

Writing $A = A_o + \Delta A$ where ΔA is the fluctuating part of the amplitude, and $\langle A\rangle = A_o \exp - \alpha x$ where α ($=\frac{1}{2}\gamma$) is the attenuation coefficient of the mean (amplitude of the) acoustic field, and provided $\alpha \ll k$, $\mu^2 \ll 1$ and $ka \gg 1$, equation (11) reduces to the parabolic equation:

$$2ik\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} + 2k^2\mu A = 0 \qquad (12)$$

Chernow has shown that this is equivalent to taking into account only the forward scattered waves.

Equation (12) can be solved by successive approximations and the first approximation gives, for the intensity attenuation coefficient of the mean acoustic field:

$$\gamma = 2\langle\mu^2\rangle k^2 \int_0^\infty N(\xi)d\xi \qquad (13)$$

For Gaussian and exponential correlation functions, this yields, respectively

$$\gamma = \sqrt{\pi}\,\langle\mu^2\rangle\,k^2 a \qquad (14a)$$

and $$\gamma = 2\langle\mu^2\rangle\,k^2 a \qquad (14b)$$

THE GREEN'S FUNCTION METHOD

Application of Green's functions in the analysis of acoustic wave propagation in random media has made multiple scattering studies feasible [9,10]. The Green's function $G(\underline{r},\underline{r}_o)$ describing an acoustic field at a point $\underline{r}$, obeys the following integral equation:

$$G(\underline{r},\underline{r}_0) = G_0(\underline{r},\underline{r}_0) - k^2\int G_0(\underline{r},\underline{r}_1)\ \epsilon(\underline{r}_1)\ G(\underline{r}_1,\underline{r}_0)\ d^3\underline{r}_1 \qquad (15)$$

where $\underline{r}_0$ is the position of the source, $\epsilon(\underline{r})$ is defined by equation (1) above, and $G_0(\underline{r},\underline{r}_0)$ is the so-called free space Green's function:

$$G_0(\underline{r},\underline{r}_0) = -\frac{\exp ik|\underline{r}-\underline{r}_0|}{4\pi|\underline{r}-\underline{r}_0|} \qquad (16)$$

that obeys the equation

$$\nabla^2 G_0(\underline{r},\underline{r}_0) + k^2\ G_0(\underline{r},\underline{r}_0) = \delta(\underline{r}-\underline{r}_0) \qquad (17)$$

Solving equation (20) by an iterative procedure and subsequently averaging, one obtains for the mean acoustic field the Dyson equation [9-11]:

$$\langle G(\underline{r},\underline{r}_0)\rangle = G_0(\underline{r},\underline{r}_0) + \iint G_0(\underline{r},\underline{r}_1) Q(\underline{r}_1,\underline{r}_2) \langle G(\underline{r}_2,\underline{r}_0)\rangle d^3\underline{r}_1 d^3\underline{r}_2 \qquad (18)$$

where Q is the mass operator (or effective wave number operator) which can be expressed in series form [9-11].

For random stationary and isotropic media, the Dyson equation can be solved [10] to give:

$$G(\underline{r}) = \frac{1}{4\pi^2 iR}\int_{-\infty}^{+\infty} \frac{Ke^{iKR}}{k^2-K^2-\frac{4\pi}{K}\int_0^\infty Q(r)\sin(Kr)r\,dr+i0}\,dK \qquad (19)$$

The effective complex wave number of the mean acoustic field is given by the poles of the integrand function [10]. Using the Bourret approximation [11] that retains only the first term of the series for Q, i.e.

$$Q(r) = k^4 G_0(r)\Psi(r)$$

where $\Psi(r)$ is the correlation function for ϵ (see equation (2)), one obtains an expression for the poles of the integrand of equation (19) as:

$$k^2-K^2 + \frac{k^4}{K}\int_0^\infty \Psi(r)e^{ikr}\sin(Kr)dr = 0 \qquad (20)$$

The effective wave number of the mean acoustic field can be shown [10] to be

$$k_{eff} = k\left[1+ \frac{\pi k}{4}\int_0^\infty K\ln\left(\frac{2k+K}{2k-K}\right)\Phi(K)dK + \frac{i\pi^2 k}{2}\int_0^{2k}\Phi(K)KdK\right] \qquad (21)$$

The imaginary part of this gives the scattering loss, so that the attenuation coefficient of the mean acoustic field is:

$$\gamma = \pi^2 k^2 \int_0^{2k}\Phi(K)KdK \qquad (22)$$

where $\Phi(K)$ is the Fourier transform of $\Psi(r)$.

Assuming a Gaussian form for $\Psi(r)$, one obtains

$$\gamma = \frac{\sqrt{\pi}<\epsilon^2>ak^2}{4}\left[1-e^{-k^2a^2}\right] \qquad (23a)$$

while for an exponential form, one obtains

$$\gamma = \frac{2<\epsilon^2>k^4a^3}{1+4k^2a^2} \qquad (23b)$$

COMPARISON OF RESULTS

The results obtained by the different methods for the two different forms of the correlation function given in equation (3) are shown in equations (5) and (8) for the mean square fluctuations, and in equations (10), (14) and (23) for the mean intensity attenuation coefficient.

The mean square phase fluctuation is equal to that of the mean square of the logarithm of the amplitude fluctuation using Rytov's analysis (equation (8)). The dependence on the parameter $D=4L/ka^2$ is of interest in that for D<<1, the results obtained are identical to those obtained by the ray theory (equation (5)), while for D>>1, the values obtained have the same dependencies, but half the magnitude.

As far as the intensity attenuation coefficients are concerned, the parabolic equation analysis (equation (14)) - as expected - gives the same results as the method of small perturbations (equation (10)) for the small wavelength limit (ka>>1). The expressions obtained for γ by the Green's function analysis (equation (23)) are identical to those obtained by the small perturbation method, provided the Bourret approximation is made. The Green's function method is clearly analytically complex, but has the significant advantage of providing a series solution to the problem so that higher order (multiple) scattering can be analysed. It appears to be a general rule that the attenuation predicted with a Gaussian correlation function is less than that with a simple exponential function.

CALCULATION OF VELOCITY DISPERSION

Miller and his co-workers [12] have suggested a local form for the Kramers Kronig relations between the frequency dependence of the attenuation coefficient and the velocity dispersion ($c(w)$) as:

$$\frac{1}{c(w)} = \frac{1}{c(w_o)} - \frac{2}{\pi}\int_{w_o}^{w} \frac{\alpha(w)}{w^2}\,dw \tag{24}$$

Equation (24) can be written in an approximate form as:

$$c(w) - c(w_o) = \frac{2c^2(w_o)}{\pi}\int_{w_o}^{w} \frac{\alpha(w)}{w^2}\,dw \tag{25}$$

Using the values of γ obtained from the Green's function analysis ($\gamma=2\alpha$), one observes that the dispersion will depend on the ratio $a:\lambda$ and also on the form of the correlation coefficient, N, of the medium irregularities. For the small wavelength limit we have ($ka>>1$):

$$c(w) - c(w_o) = B\langle\mu^2\rangle a\,\frac{c^2(w_o)}{c_o^2}\,(w-w_o)$$

and in the long wavelength limit ($ka<<1$):

$$c(w) - c(w_o) = B\,\frac{\langle\mu^2\rangle a^3}{3c_o^4}\,(w^3-w_o^3)$$

where $B = \frac{1}{\sqrt{\pi}}$ for a Gaussian correlation function in both wavelength limits

$= \frac{2}{\pi}$ for an exponential correlation, $ka>>1$

$= \frac{8}{\pi}$ for an exponential correlation $ka<<1$

Experimental confirmation of these results is still awaited.

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