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SCATTERING OF ACOUSTIC WAVES BY TURBULENCE

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INTRODUCTION

Wave propagation in random media is encountered in a large variety of the branches of physics: acoustics, biophysics, optics, radiophysics, atmospheric physics, etc. The problems associated with random media may be grouped into three main categories: discrete random scatterers, random continua, and rough surfaces [1,2]. Scattering of waves by the inhomogeneities of the medium or surface causes stochastic changes in the amplitude and phase of the waves, and - in the case of transverse waves - of the direction of polarization. A variety of mathematical methods have been employed in studies of wave propagation in random media [3-7]. The choice of the one which is actually used in a given situation, depends to a large extent on the scale of the inhomogeneities in relation to the wavelength, λ, of the radiation.

If $\lambda \ll a$ and $\lambda L \ll a^2$ (where L denotes the distance traversed by the wave and a is the mean size of the inhomogeneities), the methods of optical geometry can be used to calculate the mean square fluctuations of the time taken for a wave to reach a particular (distant) point, and also the fluctuations of phase and amplitude observed at that point.

The range of applicability of the method of small perturbations is determined by the inequalities: $\frac{\Delta A}{A_0} \ll 1$, $\Delta\phi \ll 1$, where A denotes the amplitude of a wave, and $\Delta\phi$ its phase fluctuations [7]. The method of smooth perturbations (Rytov's method [7]) has a somewhat wider range of applicability as it demands only small relative changes of amplitude and phase over the distance of a wavelength. These two conditions are fulfilled if the energy scattered during a wavelength of propagation distance is small in comparison to the initial energy of the wave.

The methods of small pertubations and of smooth perturbations have been considered in detail by Chivers [8], who has generalised Rytov's method by calculations of wave phase and amplitude fluctuations caused not only by variations in wave velocity but also in the density and the bulk modulus of the medium. The results obtained depend on the value of the parameter $D = \frac{4L}{ka^2}$ where k is the wave number, and the other notations have already been defined above.

AUTOCORRELATION FUNCTION OF FLUCTUATIONS IN THE MEDIUM REFRACTIVE INDEX FOR ACOUSTIC WAVES

The acoustical properties of random media depend not only on the value of the mean square fluctuations of the refractive index for acoustic waves, but also fluctuating parts of the square of the acoustic refractive index, we separate as follows: $n^2 = 1 + \epsilon(\vec{r})$ The autocorrelation function of the fluctuations $\epsilon(\vec{r})$ is defined by the formula:

$$\Psi(\vec{r}_1,\vec{r}_2) = \langle\epsilon(\vec{r}_1)\epsilon(\vec{r}_2)\rangle \tag{1}$$

In theoretical considerations concerning acoustic wave propagation in random media, the Gaussian form of the correlation function is often used:

$$\Psi(\vec{r}_1,\vec{r}_2) = \langle\epsilon^2\rangle e^{-\frac{(\vec{r}_1-\vec{r}_2)^2}{\xi^2}} \tag{2}$$

where $\langle\epsilon^2\rangle$ denotes the mean square fluctuation of ϵ, while ξ is the so-called radius of correlation, i.e. the mean distance over which fluctuations are correlated. The Gaussian form of the correlation function corresponds to continuous changes in the medium refractive index for acoustic waves. If changes of the medium refractive index are discontinuous, the correlation function of the medium inhomogeneities has an exponential form:

$$\Psi(\vec{r}_1,\vec{r}_2) = \langle\epsilon^2\rangle e^{-\frac{|\vec{r}_1-\vec{r}_2|}{\xi}} \tag{3}$$

as in this case [7] :

$$\left\langle\left(\frac{d\epsilon(r)}{dr}\right)^2\right\rangle = 2\lim_{r\to o}\frac{\Psi(o)-\Psi(r)}{r^2} = 2\langle\epsilon^2\rangle\lim_{r\to o}\frac{1}{r\xi} = oo.$$

One can prove [9] that for media near their critical state, the correlation function of fluctuations in the molecular concentration is given by the Ornstein-Zernike formula:

$$\Psi(r) = \frac{A}{r}\, e^{-\frac{r}{\xi}} \tag{4}$$

where $A = \frac{k_B T \beta_T}{4\pi\xi^2}$. k_B denotes the Boltzmann constant, and

$$\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T .$$

SCATTERING OF ACOUSTIC WAVES BY TURBULENCE

If inhomogeneities are caused by turbulences, correlation between medium inhomogeneities are described by the von Karman function [3,5]:

$$\Psi(r) = \langle\epsilon^2\rangle \frac{1}{2^\nu \Gamma(\nu)} \left(\frac{r}{\xi}\right)^\nu K_\nu\left(\frac{r}{\xi}\right), \tag{5}$$

where $K_\nu\left(\frac{r}{\xi}\right)$ is the Bessel function of imaginary argument, the so-called

McDonald function [10], $\Gamma(\nu)$ denotes the Euler gamma function, and ν is a number.

If the medium is statistically inhomogeneous, $\langle\epsilon^2\rangle$ and $\Psi(\vec{r}_1,\vec{r}_2)$ change with translation of the coordinate system. It is then preferable to use the so-called structural function instead of the correlation function. The structural function is defined by the equation:

$$D(\vec{r}_1,\vec{r}_2) = \langle\left[\epsilon(\vec{r}_1)-\epsilon(\vec{r}_2)\right]^2\rangle \tag{6}$$

The structural function is weakly influenced by fluctuations of great spatial extent, while the influence of these fluctuations on the correlation function is strong. Furthermore fluctuations of great spatial extent are difficult to distinguish from small changes resulting from the statistical inhomogeneity of the medium [7].

GREEN FUNCTION METHOD IN STUDIES OF ACOUSTIC WAVE PROPAGATION IN RANDOM MEDIA

Application of Green's functions in investigation of acoustic wave propagation in random media has made possible studies of the multiple scattering of waves [5,6]. Let the positions of a source and a receiver of waves be fixed by the vectors $\vec{r}_0$ and $\vec{r}$ respectively. The Green function $G(\vec{r},\vec{r}_0)$ describes the acoustic field at the point $\vec{r}$. $G(\vec{r},\vec{r}_0)$ obeys the following equation [5,11]:

$$\Delta G(\vec{r},\vec{r}_0) + k^2(1+\epsilon(\vec{r})G(\vec{r},\vec{r}_0)=\delta(\vec{r}-\vec{r}_0) \tag{7}$$

where k denotes the wave number, Δ is the Laplacian operator, and $\delta(\vec{r}-\vec{r}_0)$ the Dirac Distribution. The differential equation (7) can be written in integral equation form:

$$G(\vec{r},\vec{r}_0) = G_0(\vec{r},\vec{r}_0)-k^2\int G_0(\vec{r},\vec{r}_1)\epsilon(\vec{r}_1)G(\vec{r}_1,\vec{r}_0)d^3\vec{r}_1 , \tag{8}$$

where $G_0(\vec{r},\vec{r}_0)$ is the so called free space Green function, that obeys the equation:

$$\Delta G_0(\vec{r},\vec{r}_0)+k^2G_0(\vec{r},\vec{r}_0)=\delta(\vec{r}-\vec{r}_0) \tag{9}$$

and is given by:

$$G_o(\vec{r},\vec{r}_o) = - \frac{e^{ik|\vec{r}-\vec{r}_o|}}{4\pi|\vec{r}-\vec{r}_o|} \tag{10}$$

Solving eq. (8) by means of an iterative procedure, one obtains the following series:

$$G(\vec{r},\vec{r}_o)=G_o(\vec{r},\vec{r}_1)-k^2\int G_o(\vec{r},\vec{r}_1)\epsilon(\vec{r}_1)G_o(\vec{r}_1,\vec{r}_o)d^3\vec{r}_1 + \\ + (-k^2)^2\iint G_o(\vec{r},\vec{r}_1)\epsilon(\vec{r}_1)G_o(\vec{r}_1,\vec{r}_2)\epsilon(\vec{r}_2)G_o(\vec{r}_2,\vec{r}_o)d^3\vec{r}_1 d^3\vec{r}_2 + \ldots \tag{11}$$

from which, after averaging and introducing the so-called mass operator (the effective wave number operator) in the form:

$$Q(\vec{r}',\vec{r}'')=k^4G_o(\vec{r}',\vec{r}'')\Psi(\vec{r}',\vec{r}'') \\ +k^8\iint G_o(\vec{r}',\vec{r}_1)G_o(\vec{r}_1,\vec{r}_2)G_o(\vec{r}_2,\vec{r}'')\Psi(\vec{r}',\vec{r}_2)\Psi(\vec{r}_1,\vec{r}'')d^3\vec{r}_1 d^3\vec{r}_2 + \\ +k^8\iint G_o(\vec{r}',\vec{r}_1)G_o(\vec{r}_1,\vec{r}_2)G_o(\vec{r}_2,\vec{r}'')\Psi(\vec{r}',\vec{r}'')\Psi(\vec{r}_1,\vec{r}_2)d^3\vec{r}_1 d^3\vec{r}_2+\ldots \tag{12}$$

one obtains the Dyson integral equation:

$$\langle G(\vec{r},\vec{r}_o)\rangle=G_o(\vec{r},\vec{r}_o)+\iint G_o(\vec{r},\vec{r}_1)Q(\vec{r}_1,\vec{r}_2)\langle G(\vec{r}_2,\vec{r}_o)\rangle d^3\vec{r}_1 d^3\vec{r}_2. \tag{13}$$

The Dyson integral equation can be solved effectively for random homogeneous media (i.e. those for which the statistical characteristics $\langle\epsilon^2\rangle$ and $\Psi(\vec{r}_1,\vec{r}_2)$ do not change with translation of the coordinate system).

For random homogeneous media, G and Q are functions only of differences of the coordinates:

$$\langle G(\vec{r}-\vec{r}_o)\rangle=G_o(\vec{r}-\vec{r}_o)+\iint G_o(\vec{r}-\vec{r}_1)Q(\vec{r}_1-\vec{r}_2)\langle G(\vec{r}_2-\vec{r}_o)\rangle d^3\vec{r}_1 d^3\vec{r}_2. \tag{14}$$

Introducing the Fourier transforms of G_o, <G> and Q:

$$G_o(\vec{r}-\vec{r}_o)=\int g_o(\vec{K})e^{i\vec{K}(\vec{r}-\vec{r}_o)}d^3\vec{K} \quad ,$$

$$\langle G(\vec{r}-\vec{r}_o)\rangle=\int \langle g(\vec{K})\rangle e^{i\kappa(\vec{r}-\vec{r}_o)}d^3\vec{K} \quad , \tag{15}$$

$$Q(\vec{r}_1-\vec{r}_2)=\int q(\vec{K})e^{i\vec{K}(\vec{r}-\vec{r}_o)}d^3\vec{K} \quad ,$$

one obtains from (14) :

$$\langle g(\vec{K})\rangle=g_o(\vec{K})+(2\pi)^6 q_o(\vec{K})g(\vec{K})\langle g(\vec{K})\rangle \tag{16}$$

where:

$$g_o(\kappa = \frac{1}{(2\pi)^3(k^2-\kappa^2+i0)} \quad ,$$

as results from the equation:

$$\Delta G_o(\vec{r}-\vec{r}_o)+k^2G_o(\vec{r}-\vec{r}_o)=\delta(\vec{r}-\vec{r}_o).$$

i0 denotes an infinitesimal small imaginary number, representing absorption of the waves by the medium. Applying inverse Fourier transforms, one obtains from equation (16) :

$$\langle G(\vec{r}-\vec{r}_o)\rangle=\frac{1}{(2\pi)^3}\int \frac{e^{i\vec{K}(\vec{r}-\vec{r}_o)}}{k^2-\kappa^2-\int Q(\vec{r})\exp(i\vec{K}\vec{r})d^3\vec{r}+i0}\, d^3\vec{K} \ . \tag{17}$$

Confining ourselves to random isotropic media and introducing the spherical coordinate system, after integrating angles, the formula (17) takes the form:

$$\langle G(R)\rangle= \frac{1}{4\pi^2 iR}\int_{-\infty}^{\infty} \frac{\kappa e^{i\kappa R}}{k^2-\kappa^2-\frac{4\pi}{\kappa}\int_0^\infty Q(r)\sin(\kappa r)rdr+i0}\, d\kappa \ . \tag{18}$$

The poles of the integrand function determine the effective complex wave number of the mean acoustic field [6].

The further development requires an explicit form for the mass operator. In the so called Bourret [2] approximation:

$$Q(r)=k^4 G_0(r)\Psi(r) \tag{19}$$

the following equation determines the poles of the integrand function in formula (18):

$$k^2-\kappa^2+\frac{k^4}{\kappa}\int_0^\infty \Psi(r)e^{i\kappa r}\sin(\kappa r)dr=0\ . \tag{20}$$

In the zero order approximation $\kappa=k$, so the next first order approximation is given by the equation:

$$k^2-\kappa^2+k^3\int_0^\infty \Psi(r)e^{ikr}\sin(kr)dr=0\ . \tag{21}$$

Using the formula:

$$\left[1+k\int_0^\infty \Psi(r)e^{ikr}\sin(kr)dr\right]^{1/2} \approx 1+\frac{k}{2}\int_0^\infty \Psi(r)e^{ikr}\sin(kr)dr\ ,$$

one obtains the following expression for the effective wave number of the mean acoustic field:

$$\kappa_1=k_{ef}=k\left[1+\frac{k}{4}\int_0^\infty \sin(2kr)\Psi(r)dr+i\frac{k}{2}\int_0^\infty \sin^2(kr)\Psi(r)dr\right]\ . \tag{22}$$

It is convenient to introduce spectral decomposition of the correlation function $\Psi(r)$:

$$\Psi(r)=\iiint_{-\infty}^{\infty}\Phi(\vec{K})e^{i\vec{K}\vec{r}}d^3\vec{K}\ ,$$

or in the spherical coordinate system:

$$\Psi(r)=\frac{4\pi}{r}\int_0^\infty \Phi(k)\sin(kr)kdk \qquad (23)$$

The equation (22), after some calculations, takes the form:

$$k_{ef}=k\left[1+\frac{\pi k}{4}\int_0^\infty \kappa\ \ln\left(\frac{2k+\kappa}{2k-\kappa}\right)^2\Phi(\kappa)d\kappa + \frac{i\pi^2 k}{2}\int_0^{2k}\Phi(\kappa)\kappa d\kappa\right]. \qquad (24)$$

The scattering coefficient of the intensity of the mean acoustic field is given by the formula:

$$\gamma=2\mathrm{Im}(k_{ef})=\pi^2 k^2\int_0^{2k}\Phi(\kappa)\kappa d\kappa \quad , \qquad (25)$$

where: $\Phi(\vec{K})=(2\pi)^{-3}\iiint_{-\infty}^{\infty}\Psi(\vec{r})e^{-i\vec{K}\vec{r}}d^3\vec{r}$,

or in the spherical coordinate system:

$$\Phi(k)=\frac{1}{2\pi^2 k}\int_0^\infty \sin(kr)\Psi(r)rdr. \qquad (26)$$

PHYSICAL INTERPRETATION OF THE APPROXIMATION USED

The consideration of acoustic wave propagation in random media by means of the Green's function method can be given a visual interpretation using Feynman diagrams. The Feynman representations are as follows:

━━━━ denotes <G>

——— denotes G_0  denotes Q

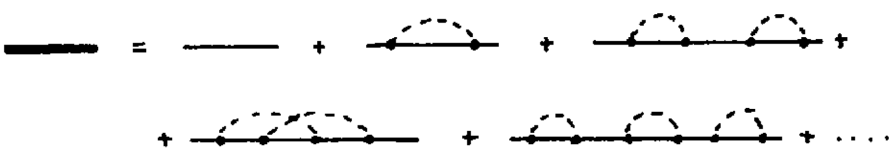
• denotes k^2

------ denotes $\Psi(r)$

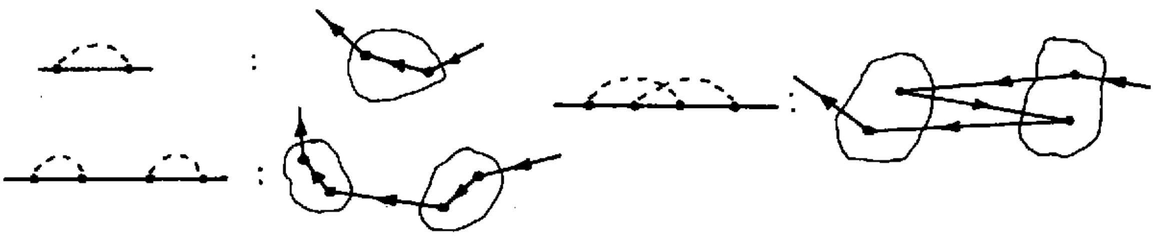
SCATTERING OF ACOUSTIC WAVES BY TURBULENCE

The Dyson (13) equation has the following representation:

or:



The following schemes of scattering correspond to the particular diagrams:



One can prove that the use of the Bourret approximation $Q(r)=k^4G_0(r)\Psi(r)$ is permissible if $k^2\xi^2\langle\epsilon^2\rangle\ll1$ [5], corresponding to fine grained random media.

CALCULATION OF THE SCATTERING COEFFICIENT FOR TURBULENT RANDOM MEDIA

For turbulent random media, the correlation function Ψ is described by the von Karman formula (5). Its Fourier transform, calculated by means of the formula (26), is:

$$\Phi(k)=\frac{\Gamma(\nu+\frac{3}{2})}{\pi^{3/2}\Gamma(\nu)}\;\frac{\langle\epsilon^2\rangle\xi^3}{(1+k^2\xi^2)^{\nu+\frac{3}{2}}}$$

Inserting this into equation (25) the value of the intensity scattering coefficient, γ, can be calculated. Figure 1 shows plots of $\frac{\gamma}{\langle\epsilon^2\rangle k}$ as a function of $k\xi$ for various values of the parameter ν. It can be seen that $\frac{\gamma}{\langle\epsilon^2\rangle k}$ increases monotonically with increasing $k\xi$ and increasing ν.

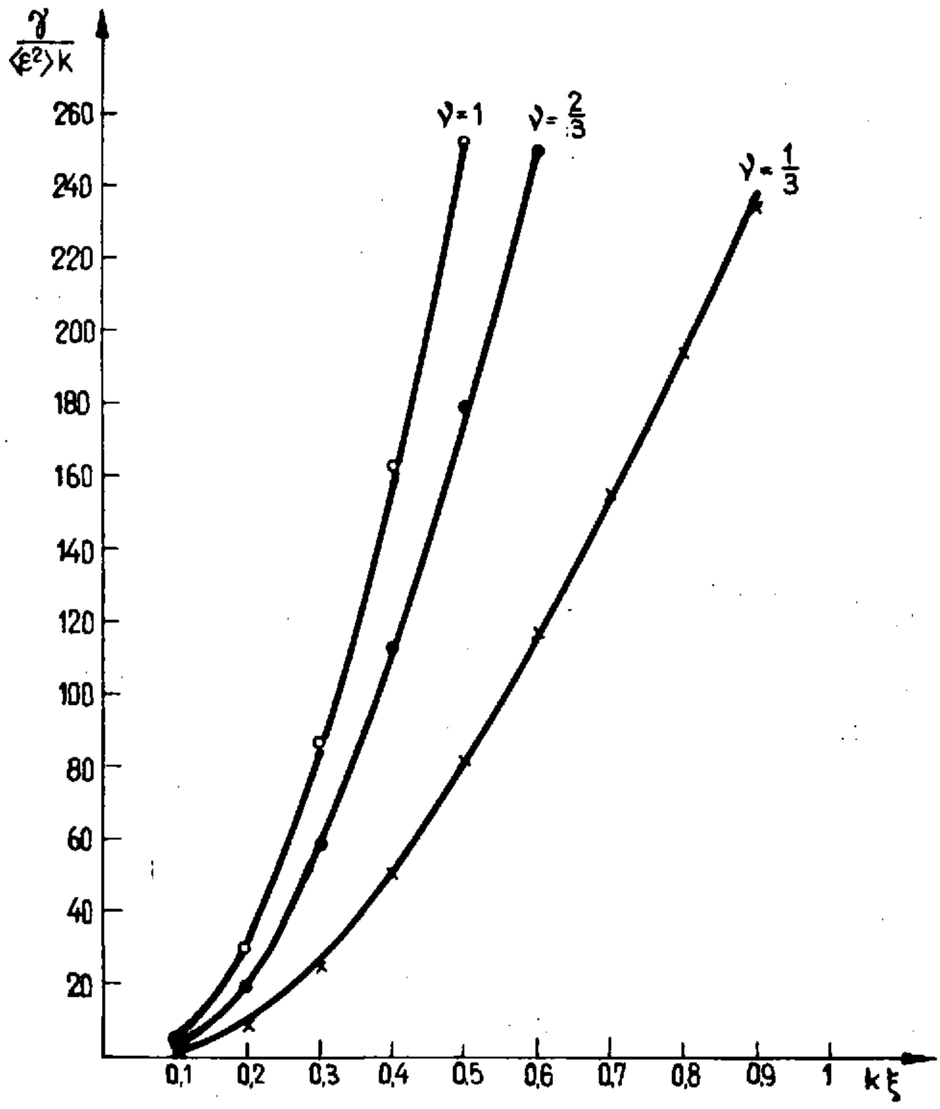


Figure 1 : Variation of the normalised intensity scattering coefficient with kξ and the parameter ν.

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