FREQUENCY SMOOTHING THROUGH CEPSTRAL AND FLOW GRAPH ANALYSIS

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1. INTRODUCTION

This paper deals with the excitation of trusses of beams with harmonic forces or displacements at nodes; efforts and displacements are measured at the same or the other nodes: two kinds of frequency transfer functionsare used : the input admittance : $Y_{ii} = \frac{v_i}{F_i}$, the transmissibility : $T_{ij} = \frac{F_i}{F_j} = \frac{v_i}{v_j}$ |1,2|

Each beam is described with the use of these two basic functions; one may write the stiffness matrix of the beam l with these functions:

$$K_l = j\omega \begin{bmatrix} Y^{-1} & Y^{-1}T \\ TY^{-1} & Y^{-1} \end{bmatrix} \qquad |3|$$

Trusses are described with the matrix stiffness assembly method: the matrix **K** is built of the individual K_l stiffness matrix of each beam |3|: $K\,|q = |F$ |4|

The aim of smoothing analysis is the calculus of a mean transfer function with few frequency points. One needs to have less variability in the transfer function according to the frequency mean used. Girard [1] calculates basic smoothed functions for the input admittance Y(f), the longitudinally $T_x(f)$ and the flexural $T_y(f)$ vibrating beam with the use of asymptotic comportment of exact functions at high frequencies:

$$T_x(f) = 1/\cos\left(\frac{2\pi fL}{c}\left(1-j\frac{\eta}{2}\right)\right) \xrightarrow{f>>} \overline{T}_x(f) = 2\ \exp -j\frac{2\pi fL}{c}\ \exp -\frac{2\pi fL\eta}{2c} \qquad |5|$$

$$Y(f) = \frac{1}{j2\pi fmc}\cot g\left(\frac{2\pi fL}{c}\left(1-j\frac{\eta}{2}\right)\right) \xrightarrow{f>>} \overline{Y}(f) = \frac{1}{j2\pi fmc} \qquad |6|$$

$$\overline{T}_y(f) = 2\ \exp -j\frac{2\pi fL}{c(f)}\ \exp -\frac{2\pi fL\eta}{4c(f)} \quad \text{with } c(f) \text{ the phase velocity of flexural waves.} \qquad |7|$$

Girard uses these basic simple functions with the stiffness assembly method to calculate transfer functions of the truss; with the assumption that non-diagonal stiffnesses vanishes from diagonal stiffnesses, formulas for the assembly are derived from the stiffness assembly matrix |4|:

$$Y_{ij} = j\omega \sum_{\text{direct structural paths}} K_{ii}^{-1} K_{iP1} K_{P1P1}^{-1} \cdots K_{PnPn}^{-1} K_{Pnj} K_{jj}^{-1} \qquad |8|$$

The sum is made over the direct structural paths of the truss: a direct structural path is a line made of beams which link successive nodes; a direct structural path never takes a node which has been already used. Two questions can be asked : Which mean is convenient with the asymptotic concept of Girard's results ? What is the meaning of the direct structural paths in the formula |4| ? Skudrzyk gives an idea for the first question: he shows [2] that the characteristic admittance, which is equal to the smoothed input admittance |6| is the geometric mean of the maxima and the minima of the exact input admittance in the case of a rod excited in longitudinal motion. It is shown [3] that equations |5,6,7| are based upon the same concept: waves in semi-infinite structures; in this paper, equation |5| is shown to be very closed to the geometric mean of the exact transfer function; the extension of this result to flexural waves in beam is also calculated. A part of the second question has been studied in reference [3]: it is seen that waves in semi-infinite media are enough to demonstrate formula |7|; in this paper, the meaning of the direct paths concept is shown with regard to loop structural paths thanks to the formalism of graph flow.

2. FREQUENCY SMOOTHING, GEOMETRIC MEAN AND CEPSTRAL ANALYSIS

2.1. Basic theory

Lets assume S(f) being a complex frequency transfer function described by its magnitude and its phase; one may calculate its time representation $s_c(t)$ by the use of the cepstral technique:

$$s_c(t) = \int_{-\infty}^{+\infty} \log|S(f)| \; \exp j2\pi ft \; df \; + \; j\int_{-\infty}^{+\infty} [\,\varphi(f) + 2k\pi\,] \; \exp j2\pi ft \; df \qquad |9|$$

Left part of expression |9| is the real part of $s_c(t)$ and is called the energy cepstra; the right part is the imaginary part of $s_c(t)$ and is calculated from a definition of the continuous phase [4]; the energy cepstra is used in the following; its main properties are related to the logarithm: a convolution of two time signals, transformed into a product in the frequency domain, becomes a sum in the cepstra; the logarithm puts the stress on low amplitude values rather than on high amplitude values; the cepstra of transmissibility diverges; with the equation |5|, we have $|T(f)| \xrightarrow[f \to \infty]{} 0$ so, $\log|T(f)| \xrightarrow[f \to \infty]{} -\infty$ and equation |9| diverges; direct and inverse numerical Fourier transforms are numerically calculated over the same complete sample of points. Cepstral analysis has been used in the characterization of acoustical reflexion coefficients of walls [5]; it has also been recently used for the reduce of variability of repetitive measurements and deconvolution of excitation [6]. Numerical results show the filtering of the cepstra in order to smooth frequency transfer functions.

2.2. Numerical results

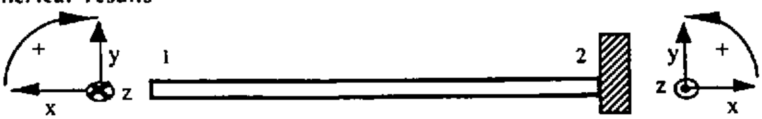

Fig. 1 : Characteristics of the beam: Young's modulus E=2.1E+10 N/m^2, shear modulus G=7.7E+11 N/m^2, bending stiffness EI=656 N.m^2, shear coefficient for a circular section γ=1.2, structural damping η=0.04, volumic mass ρ=7.8E+3 Kg/m^3, length L=1 m, section σ=0.001 m^3.

Analytical calculus have been developped for the longitudinally vibrating rod [7]; numerical results of the transmissibility are represented on figure 1 for the longitudinally and flexural vibrating beam. Their cepstra are represented respectively on figures 2 and 3.

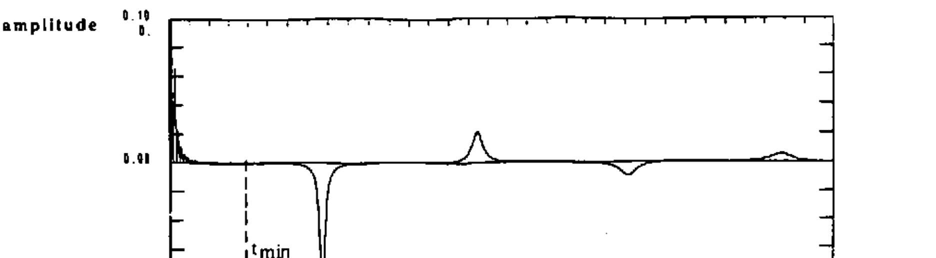

Fig. 2 : Energy cepstra of the longitudinally vibrating rod.

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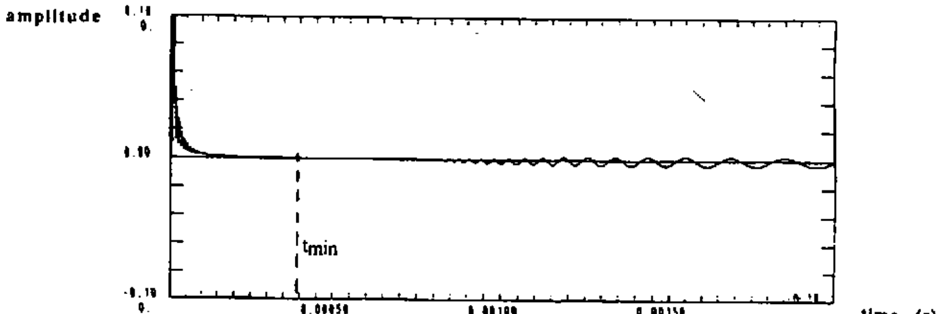

Fig. 3 : Energy cepstra of the flexural vibrating beam.

In order to put the focus on the remainder of the cepstra, the low time amplitude is truncated; zero time order tends to infinity as the frequency window grows (for cepstra response of the transmissibility diverges). Common interesting characteristics appear in figures 2 and 3; first, two time domains are revealed: in figure 2, low time amplitudes decrease quickly and alternative periodically spaced picks appear; in figure 3, the same comportment is observed, although medium time amplitudes are not periodically spaced; a good time criteria t_{min} to part these two domains is given by relation involving the length and the phase velocity of the beam; in figure 2, t_{min} takes the value: $t_{min} = 2\frac{L}{c}$. In figure 3, t_{min} is calculated with the asymptotic phase velocity of shear waves (the flexural waves in the beam are described with the rotary inertia and shear corrections [8]): $t_{min} = 2\frac{L}{c_\infty}$ with $c_\infty = \lim_{f \to \infty} c(f) = \sqrt{\frac{G}{\gamma\rho}}$. These criteria are shown on figures 2 and 3; they give important informations for cepstral filtering and its interpretation. Then, the cepstra of figures 2 and 3 is filtered in order to calculate frequency smoothed transfer functions; the first filter used is a rectangular one; its comportment is chosen in order to keep the low time domain and to eliminate the remaining time components; so, in the case of longitudinally vibrating rod (equation [5] and figure 2), the filter has the following characteristics: $F(t) = \begin{cases} 1 & \text{if } |t| < \frac{L}{c} \\ 0 & \text{if } |t| > \frac{L}{c} \end{cases}$

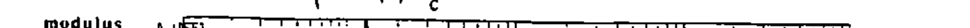

Fig. 4 : Frequency smoothing by rectangular filtering of the cepstra for the longitudinally vibrating rod.

Figure 4 shows the frequency effect of the filtering of the cepstra compared to the exact transfer function [5]; one can see that keeping the low time components of the cepstra smooths the exact transfer function over a large frequency domain. Figure 5 shows the same results for the flexural vibrating beam for which the filter has the following characteristics: $F(t) = \begin{cases} 1 & \text{if } |t| < \frac{L}{C_\infty} \\ 0 & \text{if } |t| > \frac{L}{C_\infty} \end{cases}$

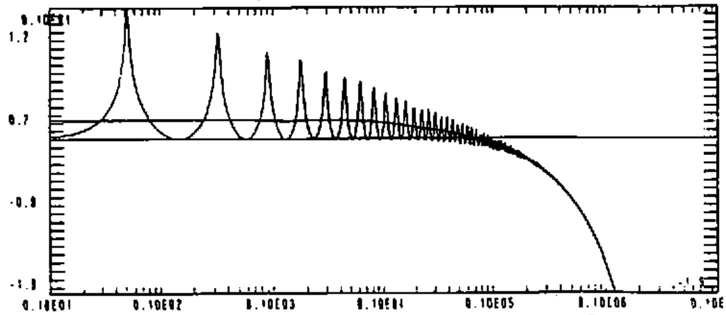

Fig. 5 : Frequency smoothing by rectangular filtering of the cepstra for the flexural vibrating rod.

Again, low time components are enough for a large frequency domain smoothing. The next filter used is a sinus cardinal filter; its interest consists in the fact that a sinus cardinal filter in the time domain corresponds to a slipping window in the frequency domain: if a sinus cardinal filter is chosen such as its zero values coincides with the periodically alternative picks of figure 2, the filter takes the following expression :

$$\text{time domain : } F(t) = \frac{\sin \frac{\pi c t}{2L}}{\frac{\pi c t}{2L}} \text{ ; frequential domain : } \widetilde{F}(f) = \begin{cases} 0 & \text{for } |f| > \frac{c}{4L} \\ \frac{2L}{c} & \text{for } |f| < \frac{c}{4L} \end{cases} \quad |10|$$

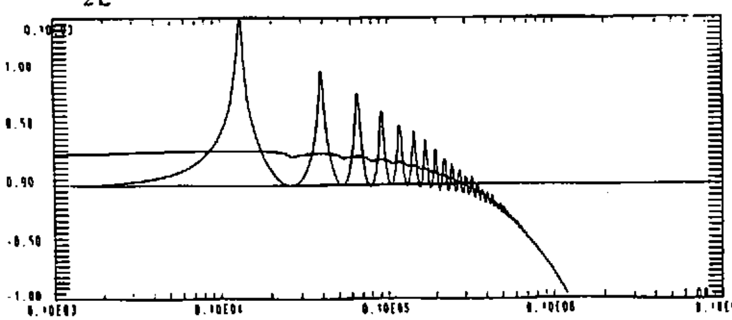

Fig. 6 : Frequency smoothing by sinus cardinal filtering of the cepstra for the longitudinally vibrating rod.

The result of this filtering is shown on figure 6; this result is also a quite good smoothing of the exact frequency function; although it is worse than the use of the rectangular filtering, it gives a direct link to the geometric mean; the filtering of the cepstra is expressed with the equation |10|:

$$\log(\overline{S}(f)) = \int_{-\infty}^{+\infty} s_d(t)\, F(t)\, \exp -j2\pi f t \, dt = \int_{-\infty}^{+\infty} \log|S(f)|\, \widetilde{F}(f-f')\, df' = \frac{2L}{c} \int_{f-\frac{c}{4L}}^{f+\frac{c}{4L}} \log|S(f')| \, df' \quad |11|$$

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It is shown in the appendix that equation |11| is the integral form of the geometric mean. The frequency smoothing shown on figure 6 for the longitudinally vibrating rod is a geometric mean of the exact transfer function through the integral transform |11|. The same kind of filter is used for the flexural vibrating beam:

$$\text{time domain}: F(t) = \frac{\sin\frac{\pi c_\infty t}{2L}}{\frac{\pi c_\infty t}{2L}} \text{ ; frequential domain}: \widetilde{F}(f) = \begin{cases} 0 & \text{for } |f| > \frac{c_\infty}{4L} \\ \frac{2L}{c_\infty} & \text{for } |f| < \frac{c_\infty}{4L} \end{cases}$$

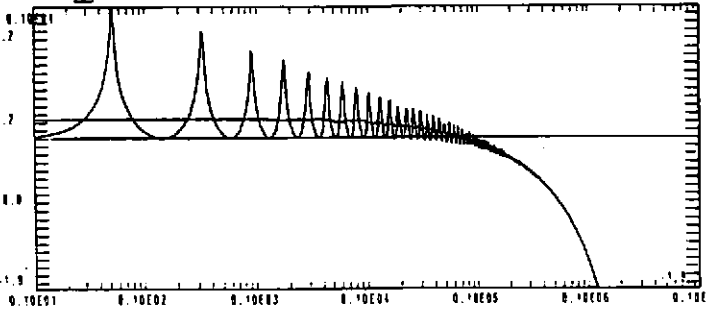


Fig. 7 : Frequency smoothing by sinus cardinal filtering of the cepstra for the flexural vibrating rod.

Results are shown on figure 7: although the aim of this filtering is different from the longitudinally vibrating rod case, the smoothed frequency function keeps its major properties in the medium and high frequency range. These results are now interpreted.

2.3. Interpretations and conclusions

The cepstra analysis of the transfer function tells about its structure; it is shown that low time filtering of the cepstral components is equivalent to asymptotic analysis and to propagation in a semi infinite medium; thus, the remaining information of the transfer function and of the cepstra only concerns the modal comportement. The term $2\Delta f$, which is the length of the integral window in the geometric mean expression (see the appendix), is equal to the modal separation (the inverse of the modal density) in the case of the longitudinally vibrating rod; this value is connected to the return time of a wave in the medium; it is shown that Girard's frequency smoothed transfer function is similar to the geometric mean in a slipping frequency window for the transmissibility. Cepstral analysis is a mean to transform transfer functions of a finite system into transfer functions of the equivalent semi infinite system in connection with the modal density of the finite system. With Skudrzyk's work about geometric mean of the input admittance, one may show that the frequency transfer function smoothed by the asymptotic concept is the geometric mean of the exact frequency function. Frequency smoothing is a deconvolution operator: one may expect to smooth experimental results by the use of the cepstral analysis and adequate filters such as the rectangular one (or the more convenient Hanning window). The main problem is to get enough information from low frequencies up to high frequencies: this implies high frequency sampling and great number of samples. Another problem is the high sensibility of the cepstral tools to noise effects.

3. FREQUENCY SMOOTHING, STRUCTURAL PATHS AND FLOW GRAPH

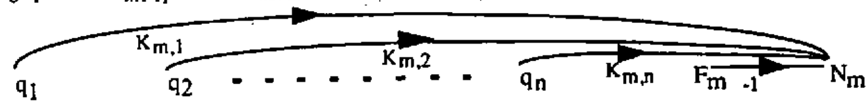
3.1. Basic theory |9|

One has to calculate the inverse of the stiffness matrix assembly |4| to solve the problem of the truss. Lets transforme this relation into a more convenient one in the flow graph theory: K |q - |F = |N |12|

where N is a vector of null variables. The flow graph theory describes each equation of this matrix relation with nodes and branches; one node corresponds to one variable; one branch links two variables through an

FREQUENCY SMOOTHING

oriented arrow according to the relation between these variables. One of these equations is associated with its graph flow: $K_{m1}q_1 + \cdots + K_{mn}q_n - F_m = N_m$ |13|



These nodes are defined in the following way: $q_1, \ldots, q_n$ are test nodes; F_m are source nodes that are the input of the graph flow; the gain is the value taken by the test node with reference to the source node. The following definitions of nodes and branches is added to complete this description: chain nodes support one convergent and one divergent branch; a loop is only composed of chain nodes; a direct path is composed of one source node, chain nodes and one test node. A transfer graph is derived from the last representation: for each node, one of its branches is inverted; many transfer graphes may be written from the last representation: the one which impose the nodes q_i to be test nodes, the nodes F_j to be source nodes is kept in the calculus of the admittance Y_{ji}. The gain of a test node with regard to a source node is calculated by the use of Mason's formula: [9]

$$Y_{ij} = j\omega \frac{\sum_n (-1)^{L_n} \Gamma_n}{1 + \sum_m (-1)^{L_m} \Gamma_m^0} \qquad |14|$$

where: Γ_n is the product of all the loop gains and the direct path gain of the $n^{ième}$ chain graph, Γ^0_m is the product of all the loop gains inside the $m^{ième}$ chain graph of the homogeneous transfer graph (where all source nodes have been suppressed), L_n is the number of loops of Γ_n, L_m is the number of loops of Γ^0_m. Mason's formula may be read in the following way: the gain is obtained by the product of transfer values of the branches that link successive nodes of the direct path and the loop. The numerator is calculated by the sum of the direct paths weighted by the value of the gain of the loop; the only contribution to the denominator is given by the loops of the homogeneous transfer graph. So, the stress is put on the direct paths and the loop contributions according to Girard's formula |8|; the next paragraph develops an application.

3.2. Application

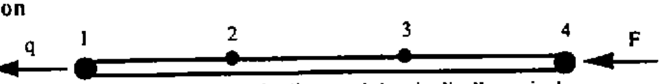


figure 8 : truss made of colinear rods longitudinally excited.

Figure 8 shows the example; it is a truss composed of four nodes and colinear rods excited at node 4 by a longitudinal effort; node 1 is the test node in the calculus of the transfer admittance: $Y_{14} = j\omega q_1 / F_4$. The stiffness assembly matrix |4| transformed into the more convenient form |12| takes the following value:

$$\begin{cases} K_{11} q_1 + K_{12} q_2 + \phantom{K_{23} q_3 +} + K_{14} q_4 - F_1 = N_1 \\ K_{21} q_1 + K_{22} q_2 + K_{23} q_3 + \phantom{K_{14} q_4} - F_2 = N_2 \\ \phantom{K_{21} q_1 +} K_{32} q_2 + K_{33} q_3 + K_{34} q_4 - F_3 = N_3 \\ K_{41} q_1 + \phantom{K_{22} q_2 +} + K_{43} q_3 + K_{44} q_4 - F_4 = N_4 \end{cases} \qquad |15|$$

Every equation of the matrix system |15| is described by a graph flow like that of equation |13|. The transfer graph is derived from the graph flow of the matrix system |15| by the consideration of the nodes q_1 q_2 q_3 q_4 as test nodes and F_1 F_2 F_3 F_4 as source nodes; the stress is also put on the input node F_4 and on the response node q_1. The deduced transfer graph is represented:

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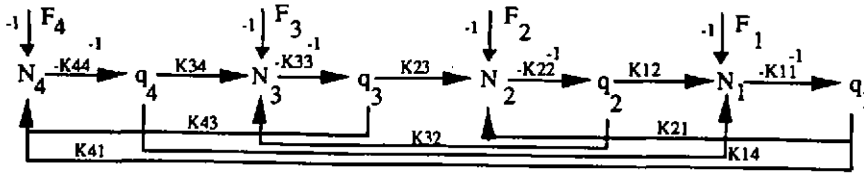

Next calculus gives all the components of Mason's formula [14]. The calculus of the numerator is done with two direct pathes $\{F_4 - N_4 - q_4 - N_3 - q_3 - N_2 - q_2 - N_1 - q_1\}$ and $\{F_4 - N_4 - q_4 - N_1 - q_1\}$ with respective gains: $K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{34}\ K_{44}^{-1}$ and $K_{11}^{-1}\ K_{14}\ K_{44}^{-1}$; only the last one is associated to a loop {N2 - q2 - N3 - q3} whose gain is $K_{23}\ K_{33}^{-1}\ K_{32}\ K_{22}^{-1}$; the denominateur is calculated with the homogeneous transfer graph (the source node F4 and its branches are suppressed); only loops contribute to the denominator: they are located and gathered in the following table:

LOOP	GAIN OF THE LOOP
$\{N_1 - q_1 - N_2 - q_2\}$	$K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{21}$
$\{N_1 - q_1 - N_4 - q_4\}$	$K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\ K_{41}$
$\{N_2 - q_2 - N_3 - q_3\}$	$K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{32}$
$\{N_3 - q_3 - N_4 - q_4\}$	$K_{33}^{-1}\ K_{34}\ K_{44}^{-1}\ K_{43}$
$\{N_1 - q_1 - N_2 - q_2 - N_3 - q_3 - N_4 - q_4\}$	$K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{34}\ K_{44}^{-1}\ K_{41}$
$\{N_1 - q_1 - N_4 - q_4 - N_3 - q_3 - N_2 - q_2\}$	$K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\ K_{43}\ K_{33}^{-1}\ K_{32}\ K_{22}^{-1}\ K_{21}$

According to the definition of Mason's formula, the transfer admittance is calculated:

$$Y_{14} = j\omega \frac{K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\left(-1 + K_{23}\ K_{33}^{-1}\ K_{32}\ K_{22}^{-1}\right) - K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{34}\ K_{44}^{-1}}{1 - \Gamma_1^0 + \Gamma_2^0} \quad [16]$$

with:
$$\Gamma_1^0 = K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{21} + K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\ K_{41} + K_{33}^{-1}\ K_{34}\ K_{44}^{-1}\ K_{43} + K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\ K_{43}\ K_{33}^{-1}\ K_{32}\ K_{22}^{-1}\ K_{21} + K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{32} + K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{34}\ K_{44}^{-1}\ K_{41}$$

$$\Gamma_2^0 = K_{11}^{-1}\ K_{12}\ K_{22}^{-1}\ K_{21}\ K_{33}^{-1}\ K_{34}\ K_{44}^{-1}\ K_{43} + K_{11}^{-1}\ K_{14}\ K_{44}^{-1}\ K_{41}\ K_{22}^{-1}\ K_{23}\ K_{33}^{-1}\ K_{32}$$

3.3. Interpretations and conclusions

The result [16] is compared to Girard's smoothed equation [8]: $\overline{Y_{14}} = j\omega \frac{\overline{K_{14}}}{\overline{K_{11}}\,\overline{K_{44}}} + j\omega \frac{\overline{K_{12}}\,\overline{K_{23}}\,\overline{K_{34}}}{\overline{K_{11}}\,\overline{K_{22}}\,\overline{K_{33}}\,\overline{K_{44}}}$

If all the loops in the equation [16] are suppressed thanks to the approximation: $K_{23}\ K^{-1}_{33}\ K_{32}\ K^{-1}_{22} << 1$, $\Gamma^0_1 << 1$, $\Gamma^0_2 << 1$, then both equations are identical; this approximation is justified at medium and high frequencies where non diagonal stiffnesses vanishe from diagonal stiffnesses in the stiffness matrix [4]. We may now interprete the expression "direct structural path" and "loop" in Girard's result [8]: the direct structural path is a flow graph path which links a source node (that is the excitation degree of freedom) to the test node (that is the response degree of freedom). It characterizes the propagation of an "open" information through the free degrees of freedom of the truss; on the other side, loops characterizes the "closed" information which causes the modal response of the truss.

4. CONCLUSIONS AND EXTENDS

By the use of two different analysis, the demonstration of two important properties for frequency smoothing has been given; first, it has been shown that geometric mean is the combined use of cepstral analysis and low time filtering (according to the modal density) to deconvolute the semi infinite part of the frequency transfer function; then, by the use of flow graph analysis, the definition of the concepts of direct structural path and loop is shown to be connected to propagation concepts; these two studies are self consistent with the hypotheses of semi inifinite systems at high frequencies. Work is in progress to experiment the cepstral analysis in order to get smoothed transfer function from mesured transfer function. This work is supported by the D.R.E.T. and INTESPACE (C.N.E.S.); acknowledgments to M. Aquilina for fructuous discussions.

5. APPENDIX

The geometric mean $\overline{H}(f)$ of a discretisation of N frequency points of the exact transfer function H(f) is defined by:

$$\overline{H}(f) = \lim_{N \to \infty} \overline{H}_N(f) = \lim_{N \to \infty} \sqrt[N]{\prod_{k=1}^{N} \left| H\left(f + 2\frac{k}{N}\Delta f\right)\right|}$$

The discrete geometric mean may be written with a logarithmic scale:

$$\log\left[\overline{H}(f)\right] = \lim_{N \to \infty} \frac{1}{N}\sum_{k=1}^{N} \log\left| H\left(f + 2\frac{k}{N}\Delta f\right)\right|$$

The sum of the last expression is transformed thanks to the Riemann integral; with a linear scale, the integrated geometric mean takes the following expression in which the parameter Δf is interpreted :

$$\overline{H}(f) = \exp\left[\frac{1}{2\Delta f}\int_{f-\Delta f}^{f+\Delta f} \log\left| H(f')\right| df'\right]$$

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