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ON KHOKHLOV-ZABOLOTSKAYA-KUZNETSOV EQUATION AND SECOND-ORDER NONLINEAR GENERATION OF THE PARAMETRIC ACOUSTIC ARRAY

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The Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation is a parabolic representation which describes the propagation of sound waves by taking into consideration diffraction, absorption and nonlinear effects. The KZK equation is considered a complete second-order model of the waves generated by the parametric acoustic array. This paper investigates the impact of diffraction on the secondary waves generated by the parametric acoustic array. For this purpose, the diffraction term of the KZK equation has been removed to form a one-dimensional equation. Simulations of both the full KZK equation and the proposed equation without diffraction have been compared for different modulation techniques. The results show good agreement between the two models in the axial propagation, so the diffraction effect can be compensated by introducing a new factor to imitate the original full KZK equation. A significant reduction in the computational cost can be achieved, as the diffraction term is the most computationally expensive part of the KZK equation. When the axial far-field propagation of the demodulated waves is analysed with different modulation techniques, the proposed abstracted equation behaves similarly to the full KZK equation.

Keywords: KZK equation, Parametric acoustic array

1. Introduction

The theory of the parametric acoustic array was developed by Westervelt in 1963 [1]. This quasi-linear approach was subsequently upgraded by Berktay in 1965 [2], by introducing a far-field solution describing a self-demodulation process in the propagation field. Since then, Berktay's solution has become one of the most important models of the parametric acoustic array. Since early experimentations of the parametric acoustic array in liquids, until the latest development of parametric loudspeakers [3, 4, 5], Berktay's solution has remained to be the model used in the implementation of the parametric acoustic array. However, only minor amendments have been done to this model [6], despite the inaccuracy of the equation describing the parametric acoustic array.

On the other hand, the Khokhlov - Zabolotskaya - Kuznetsov (KZK) equation has been developed [7], from the basis of the KZ equation [8], and resulted to be a more accurate description of the parametric acoustic array [5]. Due to its complexity, KZK equation does not have an analytical solution, like Berktay's equation describing the self-demodulation process. The KZK equation takes into consideration absorption, nonlinearity and diffraction, which allows higher accuracy, but it also requires more computational resources, especially for the diffraction effect. Despite the existence of investigation on the diffraction term in the KZK equation [9, 10], there is a lack of research on the effect of diffraction in the generation of the parametric acoustic array. This paper investigates the contribution of the diffraction term in the generation of secondary waves in the parametric acoustic array.

The rest of the paper is structured as follows: Section 2 describes the KZK equation, Berktay's solution and the generation of secondary waves, Section 3 describes the simulations performed, Section 4 shows the results, and Section 5 contains the conclusions and future work.

2. KZK equation

The Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation describes the combined effects of diffraction, absorption and nonlinearity present in the directional sound beams.

If we consider z as the propagation direction, xy is the plane perpendicular to the propagation axis, the source with radius a is located at $z = 0$, and it radiates at frequencies satisfying $ka >> 1$ [7], where k is the wavenumber. If the three effects (diffraction, absorption and nonlinearity) are arranged to be of order $O(\epsilon^2)$ and the higher order terms are neglected, the KZK equation can be expressed in the form:

$$\frac{\partial^2 p}{\partial z \partial t'} - \frac{c_0}{2}\nabla^2 p - \frac{\delta}{2{c_0}^3}\frac{\partial^3 p}{\partial t'^3} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial t'^2} \tag{1}$$

Where p is the sound pressure, z is the propagation axis, $t' = t - z/c_0$ is a retarded time variable, c_0 is the small signal sound speed, δ is the sound diffusivity, ρ_0 is the ambient density, x and y are the plane coordinates along the propagation axis, and β is the coefficient of nonlinearity.

$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the Laplacian which operates in the plane xy and it corresponds to the diffraction term. The third term on the left hand side corresponds to absorption, and the nonlinearity term is on the right hand side.

In order to calculate the KZK equation, a time-domain algorithm, called the Texas Code [11], has been chosen. This algorithm is based on the cylindrical form of KZK equation:

$$\frac{\partial p}{\partial z} = \frac{c_0}{2}\int_{-\infty}^{t'}\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right)dt'' + \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial t'^2} + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial t'} \tag{2}$$

Where $r = \sqrt{x^2 + y^2}$ is the radial distance from the center of the beam. Diffraction is the first term on the right hand side, absorption is the second one, and nonlinearity is the third term.

The Texas Code calculates diffraction, absorption and nonlinearity separately for each propagation point ∂z. As the purpose of this research is to study the effect of diffraction on the parametric acoustic array, the diffraction term has been removed from KZK equation. When the diffraction term is discarded, Eq. (2) becomes:

$$\frac{\partial p}{\partial z} = \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial t'^2} + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial t'} \tag{3}$$

Which has the form of a one-dimensional planar waves Burgers equation.

Texas Code uses transformed dimensionless variables [12], in which the propagating coordinates are expressed by:

$$\sigma = \frac{z}{z_0} \tag{4}$$

This means each unit in the propagation axis corresponds to a Rayleigh distance unit. A further description of the KZK texas algorithm can be found in [11, 12, 13].

Table 1: Envelopes of modulation techniques used.

Modulation Technique	Envelope
DSBAM	$1 + mg(\tau)$
SRAM	$\sqrt{1 + mg(\tau)}$
MAM1	$\sqrt{1 + mg(\tau) + \frac{1}{8}m^4 g^4(\tau)}$
MAM2	$\sqrt{1 + mg(\tau) + \frac{1}{16}m^6 g^6(\tau) + \frac{1}{128}m^8 g^8(\tau)}$
MAM3	$\sqrt{1 + mg(\tau) + \frac{5}{128}m^8 g^8(\tau) + \frac{1}{128}m^{10} g^{10}(\tau) + \frac{1}{512}m^{12} g^{12}(\tau)}$

2.1 Berktay's equation

Berktay developed a closed solution describing the far-field behaviour of the secondary waves produced by the parametric acoustic array [2]. This solution is formulated by:

$$p_2 = \frac{\beta p_0^2 a^2}{16 \rho_0 c_0^4 z \alpha_0} \frac{\partial^2}{\partial \tau} E^2(\tau) \tag{5}$$

Where p_0 is the amplitude of the original primary signal, a is the radius of the source, α_0 is the absorption coefficient of the mean frequency $\omega_0 = \frac{\omega_1 + \omega_2}{2}$ and $E(\tau)$ is the envelope of the original primary signal $p_1 = p_0 e^{-\alpha_0 z} E(\tau) \sin(\omega_0 \tau)$. ω_1 and ω_2 are the primary frequencies of the sound beam.

Berktay's equation suggests the secondary wave generated by the parametric acoustic array is a demodulated wave from the original signal. Because of this assumption, the implementations of the parametric acoustic array include a modulation stage, based on Berktay's equation.

3. Simulations setup

The KZK equation has been calculated for a parametric acoustic Gaussian source, projecting a 1 kHz wave modulating a 60 kHz carrier frequency, with an amplitude of 20 Pa. The source is circular, with a radius of 22.5 cm, and the medium considered is air. For this purpose, the KZK Texas Code has been used, as described in Section 2.

To study the impact of diffraction in the generation of secondary waves by the parametric acoustic array, the KZK equation was calculated with and without the diffraction term, in the forms of Eqs. (2) and (4), respectively.

Some of the most common modulation techniques in parametric sources were selected [4, 5, 14]. These are: Double Side-band Amplitude modulation (DSBAM), Square-root Amplitude modulation (SRAM) and Modified Amplitude modulation (MAM), as indicated in Table 1.

Different modulation indices (m) were used in addition to different modulation techniques, in order to observe the impact of these parameters upon the diffraction effect. DSBAM simulations were implemented with modulation indices from 0.3 to 1.0. SRAM, MAM1, MAM2 and MAM3 simulations were implemented with a fixed modulation index of 0.5.

For all cases, 3 sets of simulations were adopted. The original KZK equation; as presented in Eq. (2), KZK equation without diffraction; as shown in Eq. (4), and Berktay's far-field solution; as shown in Eq. (5).

All simulations were implemented on the New Zealand eScience Infrastructure (NeSI) High Performance computing facilities. All results presented in the next section are axial values calculated on the propagation axis z and on the range $0 < \sigma < 100$, as described by Eq. (4).

4. Results

The acoustic field was calculated on the propagation axis for a 1 kHz demodulated secondary wave and its first two harmonics, corresponding to 2 and 3 kHz. For convenience, the best fitted curves of the full KZK equation in Fig. 1 and 2 were plotted. The proposed equation and Berktay's curves are the actual results. Fig. 1 shows the demodulated secondary waves produced by DSBAM with modulation index of 0.5:

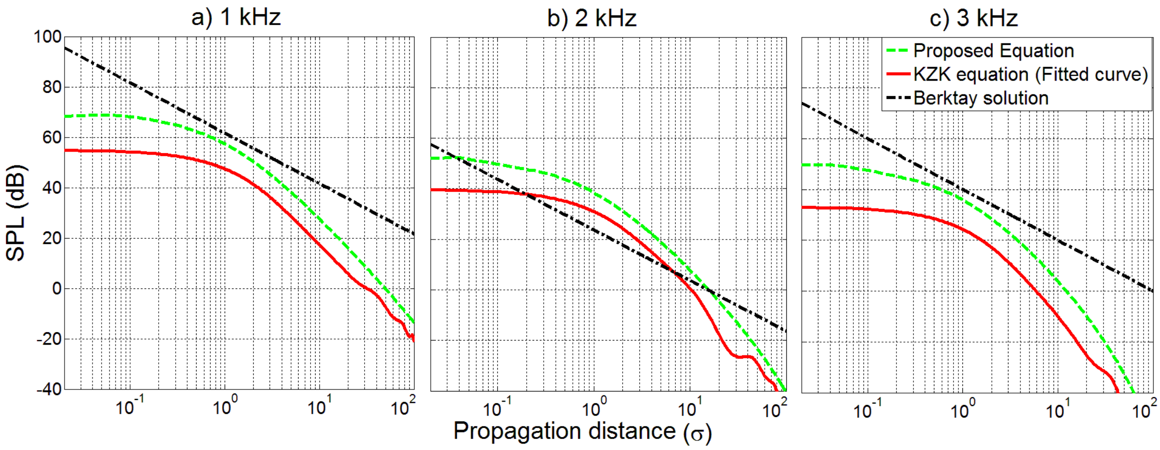

Figure 1: Demodulated secondary waves of DSBAM simulation: a) 1 kHz, b) 2 kHz and c) 3 kHz.

Both the full KZK equation and the proposed equation without the diffraction term, show almost similar performance, with a difference in the overall level. In the very far-field, both curves show some differences. On the other hand, Berktay's solution remains linear all over the propagation distance.

Figure 2 shows the demodulated secondary waves produced by SRAM, with modulation index 0.5:

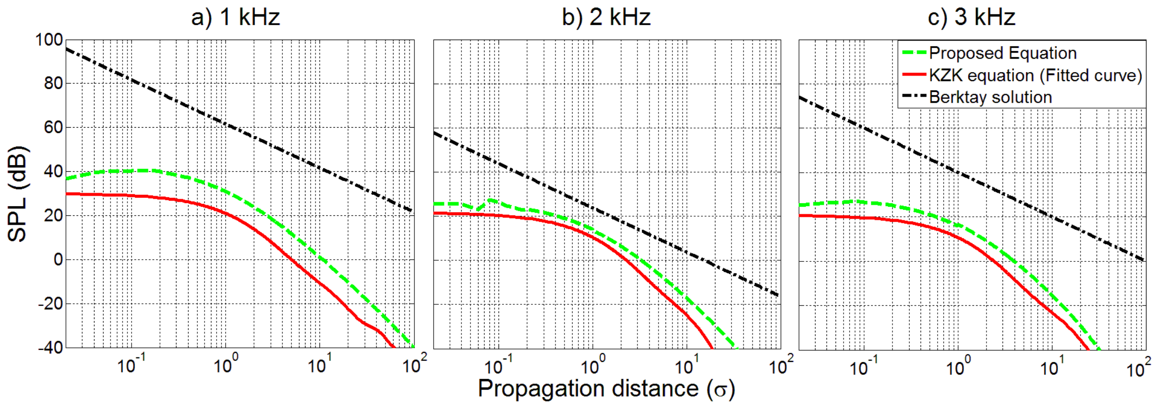

Figure 2: Demodulated secondary waves of SRAM simulation: a) 1 kHz, b) 2 kHz and c) 3 kHz.

The curves of the original KZK results are very similar to the curves of the proposed equation, similarly to the results shown in Fig. 1.

A correction factor should be applied to match the results of the proposed equation to the full KZK equation. For this purpose, the mean difference of both equations of the 1 kHz frequency demodulated wave was calculated in the range of $1 < \sigma < 10$. This range is selected, as it is a suitable range of operation for a parametric loudspeaker. The correction factor proposed mostly compensates the difference in performance between the two equations. The results calculated for the DSBAM simulations are shown in Table 2, and the rest of the modulation techniques in Table 3.

Table 2: Mean differences of DSBAM simulations

DSBAM		
Modulation Index	Mean Difference (dB)	Correction factor
0.3	10	0.316228
0.4	9.7971	0.323702
0.5	9.7517	0.325398
0.6	9.4947	0.33517
0.7	9.0773	0.35167
0.8	8.8515	0.360932
0.9	8.5558	0.373431
1	8.2778	0.385576

Table 3: Mean differences of other Modulation techniques

Modulation Technique	Mean Difference (dB)	Correction factor
SRAM	11.2296	0.274486
MAM1	12.5652	0.235364
MAM2	17.3315	0.135964
MAM3	7.8587	0.404636

The results in Table 2 show the required correction depends on the modulation index, yet not very significantly. When the modulation index is increased, the correction required decreases, by less than 2 dB, from the minimum to the maximum modulation index tested (0.3 to 1.0). As shown in Figs. 1 and 2, the KZK equation predicts a lower level than the proposed equation, so the correction factor calculated in all cases must be less than 1. Table 3 shows different modulation indices require different correction factors, as the difference could be as little as 7.8 dB, like the case of MAM3, or it could be as large as 17.3 dB, as in the case of MAM2.

In all cases in Tables 2 and 3, the correction is valid in the far-field, in the range of $1 < \sigma < 10$. The curve of the proposed equation becomes different to the KZK equation in the very far-field, when the range of $\sigma > 20$ is exceeded, as shown in Figs. 1 and 2. This difference can be explained by the fact that the sound beam produced by the parametric acoustic array becomes a spherical wave in the far-field. For this reason, the diffraction term starts to be important in this region, in opposition to the closer field behaviour, where the beam behaves as a planar wave.

5. Conclusions

The parametric acoustic array has been successfully simulated. A Burgers equation with no diffraction term, has been presented. The proposed equation requires a correction factor to match the KZK equation results, and it has been proven to be accurate on the propagation axis. The correction factor has been considered from the fundamental demodulated wave, and it has not taken into consideration the harmonics of this frequency. The correction coefficients provided correspond to the specific cases presented, as different modulation techniques, modulation indices or medium parameters will generate different correction factors. Although Figs. 1 and 2 show good agreement of all three demodulated waves, future work should include an analysis of these harmonics to quantify the harmonic distortion.

The proposed equation presents a simple solution to obtain the demodulated secondary waves. As the diffraction term is the most computationally expensive part of the KZK algorithm, this equation is much faster to be calculated. Additionally, if compared to Berktay's far-field equation, this equation shows more accurate results in a larger range, as it does not rely in Berktay's approximations.

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