ACTUAL BEARING MEASUREMENT ACCURACY OF A HR METHOD AND ESTIMATION

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1. INTRODUCTION

Target tracking and TMA batch algorithms use criterions based on the statistical likehood of the measurements obtained through array processing. The computation of this likehood requires to know the measurements distribution or simply the measurements precision (gaussian case).The purpose of this paper is to examine some practical problems arising from the use of an "orthogonal subspace method" as array processing. Four methods to evaluate bearing accuracy are presented. One of them seems to have some practical advantages and appears as a theoretical curiousity.

2.CONTEXT

2.1 ASSUMPTIONS, NOTATIONS:

To simplify we assume that the array is a "well sampled" (Shannon) uniform linear array (ULA) (N sensors) referenced to mid-point. Targets are in the array far field and propagation is kind enough so that a target induces at the array a steering vector of the form:

$$\underline{d}_\theta=\underline{d}(\theta)=\underline{d}(f,\theta)=\left[\text{cexp}(2i\pi\,(n-\frac{N+1}{2})\,f\sin\theta)\right]_{n=1,N} \qquad (0)$$

with f=F d/c normalised frequency, F frequency, d distance beetween two sensors, c sound velocity, θ bearing, $i^2=-1$, cexp(x) = complex exponential function.

We assume that signals are stationnary and we can get an unbiased estimate $\widehat{\Gamma}$ (periodogram) of the interspectral matrix Γ which is of the form: $\Gamma=\sum_{i=1}^{P} S_i\,\underline{d}_i\,\underline{d}_i^* + \mathbf{Id}$ (1)

P = number of sources, N = number of sensors, K = number of snapshots used to estimate $\widehat{\Gamma}$, $\Delta\Gamma=\widehat{\Gamma}-\Gamma$, **Id** = identity matrix, θ_i (i=1,...P) = bearings of the P sources, S_i= signal to noise ratios of the P sources, $\underline{d}_i=\underline{d}(\theta_i)$, $\delta_\theta=\frac{1}{N}\underline{d}_\theta\underline{d}_\theta^*$. $\lambda_i,\underline{u}_i$ (i=1,...,N) = eigenvalues and normalized eigenvectors of Γ in decreasing order. $\widehat{\lambda}_i$, $\widehat{\underline{u}}_i$ (i=1,...,N) = eigenvalues and normalized eigenvectors of $\widehat{\Gamma}$ in decreasing order, $\bar{\lambda}=\frac{1}{N-P}\sum_{i=P+1}^{N}\widehat{\lambda}_i$, $\mathbf{M}=\sum_{i=1}^{P}\frac{1}{\lambda_i-1}\underline{u}_i\underline{u}_i^*$, $\Pi=\sum_{i=1}^{P}\underline{u}_i\underline{u}_i^*$ (matrix of the signal subspace projector), $\Pi^\perp=\mathbf{Id}-\Pi$, $\widehat{\Pi}=\sum_{i=1}^{P}\widehat{\underline{u}}_i\widehat{\underline{u}}_i^*=\Pi+\delta\Pi+\delta^2\Pi+\ldots$, $\delta\Pi$ and $\delta^2\Pi$ first and second

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order developpement (in $\Delta\Gamma$) of $\hat{\Pi}$. $2\theta_3$ = Cramer-Rao angle (see [3]). "Tr( )" is the Trace operator, "E" is the mathematical expectation, "*" indicate transposed-conjugated, "(n)" indicate the nth derivate relative to θ. "$\alpha \chi^2_Q$ is a chi-square random law with Q degrees of freedom.

Remark: Assumptions (0) and (1) can be extended (except for the expressions of the Cramer-Rao bound given in the sequel) to the more general case where $\underline{d}_\theta$ has any form and $\Gamma = \sum_{i=1}^{P} \gamma_i \underline{d}_i \underline{d}_i^* + \mathbf{R_b}$ where γ_i are the powers of the P sources and $\mathbf{R_b}$ the known noise interspectral matrix.

2.2 ARRAY PROCESSING:

Array processing is often performed by the research of the maximums (or minimums) of a pecular function $\hat{g}$. In narrow band case this function is depends both on θ (= bearing) and F (F=frequency). In wide band case it is a function of θ and B (B=frequency band). Although the behaviour with F (or B) can be of great interest for detection and localization, we only consider here the problem of spatial narrow band localisation. That is: we fix a frequency F (we suppose that all interesting targets are detectable at the frequency F and that their narrow band spectrums at F are stable). Many array processings $\hat{g}(\theta)$ can be written:

$\hat{g}(\theta)=g(\theta,\hat{A})=\frac{1}{N}\underline{d}_\theta^* \hat{A}\, \underline{d}_\theta = Tr(\hat{A}\, \delta_\theta)$ (2a)(2b) where $\hat{A}$ is an estimate of a matrix **A**.

(I) classical beamforming $\mathbf{A} = \Gamma$ (II) adaptative beamforming $\mathbf{A} = \Gamma^{-1}$ (III) goniometer $\mathbf{A} = \Pi^{\perp}$

The "asymptotical" expression of $\hat{g}(\theta)$ is $g(\theta) = Tr(A\delta_\theta)$ (2c)

The bearings θ_q (q=1,P) of the P sources are estimated by the arguments of the maximums (I) or minimums (II) (III) of $\hat{g}(\theta)$: $\forall$ q=1,...,P $\hat{\theta}_q$= arg(Maxormin $\hat{g}(\theta)$) (3)

For a single source (P=1) we have: θ_1=arg(Maxormin $g(\theta)$) for (I) (II) (III) but in the general case (P>1) the relation: $\forall$ q=1,...,P θ_q = arg(Maxormin $g(\theta)$) (4) is only true for (III).

We introduce here the notation: $\Delta\theta_q = \hat{\theta}_q - \theta_q$

2.3 TRADITIONAL MEASUREMENTS DISTRIBUTIONS:

2.3.1 Law of $\hat{\theta}$

$\hat{\theta}$ is often assumed to have a gaussian distribution. We do not give here any theorical justification but will empirically test this assumption in the pecular considered case.

For (I) (II) (III) in the case of a single source, the estimator of θ_q reaches asymptotically (that is when K tends to ∞.) the Cramer-Rao bound and becomes efficient. That is the reason why one often considers in the general case (P>1): bias($\hat{\theta}_q$)=E $\Delta\theta_q$=0 (5) , $\sigma_\theta^2 \stackrel{\Delta}{=} E\,(\Delta\theta_q)^2 = I^{-1}(\theta)$ (6)

That is when needed σ_θ is approximated by the Cramer_Rao (CR) bound for a single source or the Woodward formula (Cramer-Rao bound with deterministic signal [3]).

Cramer_Rao bound (random signal) $\sigma_{\theta CR}=\dfrac{\theta_3\sqrt{1+NS_q}}{\sqrt{K}\,N\,S_q}$ (7)

Because (4) is only true for (III), (5) is not true even asymptotically for (I) and (I I) in the general case. Conversely it can be prooved (see [6] and [1]) that (III) verifies (5) asymptotically.

2.3.2 Law of $\hat{g}$..

$\hat{g}(\theta)$ has a distribution which is asymptotically known for (I) and (II):

(I): $\hat{g}(\theta)$ follows a $\alpha\chi^2_{2K}$ at any fixed θ. (II): $1/\hat{g}(\theta)$ follows a $\alpha\chi^2_{2K-2N+2}$ at any fixed θ. ([2])

In the sequel, we will consider (III) and its pecular problems.

2.3.3 Power estimation, usual bearing precision estimation:

Because $g(\theta_q)$ gives for (I) and (II) the signal to noise ratio S_q of the source at bearing θ_q (accurately in the case of a single source, approximately in general cases. For (I) $g(\theta_1)=1+NS_1$, for (II) $1/g(\theta_1)=1+NS_1$), $\hat{g}(\hat{\theta}_q)$ gives an estimate of the power. So it is not difficult to evaluate S_q and so $\sigma_{\theta CR}$. (e.g. $\widehat{S_q}=\frac{1}{N}(\hat{g}(\theta_q)-1)$ for (I) and $\widehat{\sigma_{\theta CR}}=\dfrac{\theta_3\sqrt{1+N\widehat{S_q}}}{\sqrt{K}\,N\,\widehat{S_q}}$). For (III) the estimation of $\sigma_{\theta CR}$ becomes more difficult because relation beetween S_q and $g(\theta_q)$ is more complicated.

3 PECULAR CASE OF (III)

It is assumed that the number of sources is exactly known.

3.1 distribution of $\hat{\theta}_q$

As usual we assume $\hat{\theta}_q$ being gaussian and test here this hypothesis with the Kolmogorov test of fit.

Simul1:

Conditions of simulation are P=1, N=16 (ULA), f=0.380 , θ_1 = 5 degrees and various (K,S_1).10000 independant measurings have been processed and the test of fit was processed over 250 groups of 40 samples. Mean and variances of $\hat{\theta}_q$ are estimated over the 10000 samples.

S_1=	0.1	0.1	1.0	1.0
K=	50	500	50	500
Probability of false rejection (%)=	5%	5%	5%	5%
% of rejection=	4.2%	4.4%	5.6%	5.2%

Simul2:

Conditions of simulation are P=3, N=16 (ULA), f=0.380, θ_1 = 5, θ_2 = -5, θ_3 = -10 degrees, S_1=0.1,

S_2=1.0, S_3=1.0, K=500, test only $\hat{\theta}_1$ over 10000 measurings (250*40):

Probability of false rejection: 5% , Observed rejection: 5.6%

So we can assume $\hat{\theta}_q$ is following a gaussian law.

3.2 asymptotical distribution of $\hat{g}(\hat{\theta}_q)$:

A second order developpement of $\hat{g}(\theta)$ near θ_q gives the value of its maximum:

$$\hat{g}(\hat{\theta}_q) = -Tr(\delta^2 \Pi\, \delta_\theta) - \frac{\left(Tr(\delta \Pi\, \delta^{(1)}_{\theta_q})\right)^2}{2\, Tr(\Pi^{\perp}\, \delta^{(2)}_{\theta_q})} \quad (10)$$

In [1] $\delta\Pi$ and $\delta^2\Pi$ have been calculated. So:

$$\hat{g}(\hat{\theta}_q) = Tr(\mathbf{M}\, \Delta\Gamma\, \Pi^{\perp} \Delta\Gamma\, \mathbf{M}\, \delta_\theta) - \frac{2}{\alpha_m N} \left(REAL\{\, Tr(\mathbf{M}\, \Delta\Gamma\, \Pi^{\perp} \underline{d}^{(1)}_q \underline{d}^{*}_q\,)\}\right)^2 \quad (11)$$

and $\hat{g}(\theta_q) = Tr(\mathbf{M}\, \Delta\Gamma\, \Pi^{\perp} \Delta\Gamma\, \mathbf{M}\, \delta_\theta)$ (12)

In the case of a single source, the law of $\hat{g}(\theta_q)$ is $\dfrac{\lambda_1}{2\,(\lambda_1 - 1)^2 K}\, \chi^2_{2N-2}$ (13)

Empirically the correlation coefficient between $\hat{g}(\theta_q)$ and $\hat{g}(\hat{\theta}_q)$ is close to 1.

We assume $\hat{g}(\hat{\theta}_q)$ follows an $\alpha\, \chi^2_Q$ low ($Q \approx 2(N-P)$) (14) and test this hypothesis (same conditions § 3.1)

Simul1: P=1

S_1=	0.1	0.1	1.0	1.0
K=	50	500	50	500
Probability of false rejection (%)=	5%	5%	5%	5%
% of rejection=	5.2%	5.2%	6.8%	6.0%
Q (2(N-P)=30) ≈	15.1	26.4	20.9	28.4

Simul2: P=3, S_1=0.1, K=500

Probability of false rejection: 5% , Observed rejection: 4.8% (Q=21.5) (2(N-P)=26)

3.3 Power estimation:

A solution is to estimate S_i by means (see [4]) of the formula:

$$\widehat{S}_q = \left(\sum_{i=1}^{P} \frac{\left| \underline{d}(\hat{\theta}_q)^{*}\, \hat{\underline{u}}_i \right|^2}{\hat{\lambda}_i - \bar{\lambda}} \right)^{-1} \quad (8) \qquad \text{because: } S_q = \left(\sum_{i=1}^{P} \frac{\left| \underline{d}(\theta_q)^{*}\, \underline{u}_i \right|^2}{\lambda_i - 1} \right)^{-1} \quad (9)$$

This estimation overestimates the power of the sources, and the evaluation of the CR bound would be underestimated (see § 3.5).

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3.4 Bearing precision:

In [1] Mr Foster gives for (III) the true variance and bias of the estimate $\hat{\theta}_q$ of bearing θ_q which he defines as: $\mathrm{var}(\hat{\theta}_q)=E\,(\Delta\theta_q)^2$ (14) . $\mathrm{bias}(\hat{\theta}_q)=E\,\Delta\theta_q$ (15)

We observe here that $\mathrm{var}(\hat{\theta}_q)$ is not really a variance which should be defined as:

$E\,(\hat{\theta}_q-E(\hat{\theta}_q))^2 = \mathrm{var}(\hat{\theta}_q) - \mathrm{bias}^2(\hat{\theta}_q)$ (16)

The justification of this definition is that bias/var$^{1/2}$ tends to 0 asymptotically.

We are only interested in $\mathrm{var}(\hat{\theta}_q)$ which is $\mathrm{var}(\hat{\theta}_q) = \frac{1}{2K\alpha_q}\sum_{i=1}^{P}\frac{\lambda_i}{(\lambda_i-1)^2 N}\,|\,\underline{u}_i^*\underline{d}(\theta_q)\,|^2$ (17)

with $\alpha_q=\|\,\frac{1}{N}\,\Pi^{\perp}\underline{d}_q^{(1)}\,\|^2$ $\quad \underline{d}_q^{(1)} = \left(\frac{\partial}{\partial\theta}\underline{d}(\theta)\right)_{\theta=\theta_q}$

We notice that: $E\,\hat{g}(\theta_q) = \frac{N-P}{K}\sum_{i=1}^{P}\frac{\lambda_i}{(\lambda_i-1)^2 N}\,|\,\underline{u}_i^*\underline{d}(\theta_q)\,|^2$ (18)

and so: $\mathrm{var}(\hat{\theta}_q) = \frac{1}{2\,(N-P)\,\alpha_q}\,E\,\hat{g}(\theta_q)$ (19)

3.5 Estimation of bearing precision:

Using the previous relations, some estimators of the variance of bearings $\mathrm{var}(\hat{\theta}_q)$ can be derived:

(i) empirical estimation: $\widehat{\mathrm{var1}} = \frac{1}{L-1}\sum_{l=1}^{L}(\hat{\theta}_{[l]} - \frac{1}{L}\sum_{l=1}^{L}\hat{\theta}_{[l]})^2$ (20)

[l] indicate a temporal indice, the number of the source is not indicated to simplifie notations.

(ii) CR formula: $\widehat{\mathrm{var2}} = \frac{1}{L}\sum_{l=1}^{L}\widehat{\sigma^2_{CR[l]}} = \frac{1}{L}\sum_{l=1}^{L}\left(\frac{\theta_3\sqrt{1+\widehat{NS_{[l]}}}}{\sqrt{K}\,N\,\widehat{S_{[l]}}}\right)^2$ (21)

(iii) true values: $\widehat{\mathrm{var3}} = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{2\,(N-P)\,\widehat{\alpha_{[l]}}}\sum_{i=1}^{P}\frac{\hat{\lambda}_i\,\bar{\lambda}}{(\hat{\lambda}_i-\bar{\lambda})^2 N}\,|\,\hat{\underline{u}}_i^*\underline{d}(\hat{\theta}_{[l]})\,|^2$ (22) derived from (17)

(iiii) true values: $\widehat{\mathrm{var4}} = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{2\,(N-P)\,\widehat{\alpha_{[l]}}}\,\hat{g}(\hat{\theta}_{[l]})$ (23) derived from (18)

with $\widehat{\alpha_{[l]}} = \|\,\frac{1}{N}\,\widehat{\Pi^{\perp}}\,\widehat{\underline{d}_{[l]}^{(1)}}\,\|^2$ $\quad \widehat{\underline{d}_{[l]}^{(1)}} = \left(\frac{\partial}{\partial\theta}\underline{d}(\theta)\right)_{\theta=\hat{\theta}_{[l]}}$

Comparison of these estimators: Empirically we test these estimators over 10000 samples:

Simul1: P=1

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S_1=	0.1	0.1	1.0	1.0
K=	50	500	50	500
$\widehat{var1}$	0.344	0.0288	0.0185	0.00184
$\widehat{var2}$	0.133	0.0258	0.0178	0.00181
$\widehat{var3}$	0.141	0.0260	0.0174	0.00181
$\widehat{var4}$	0.232	0.0265	0.0176	0.00176
CR bound	0.277	0.0277	0.0181	0.00181
corr(*)	0.23 ±0.03	0	> 0.	0

(*) correlation between $\hat{\theta}_q$ and $\hat{g}(\hat{\theta}_q)$

Simul2: P=3, S_1=0.1, K=500

$\widehat{var1}$=0.0769 $\widehat{var2}$=0.0260 $\widehat{var3}$=0.0624 $\widehat{var4}$=0.0635 CR bound=0.0277

Exemple: case (P, S_1, K)=(1, 0.1, 50) evolution of the estimates versus number of samples:

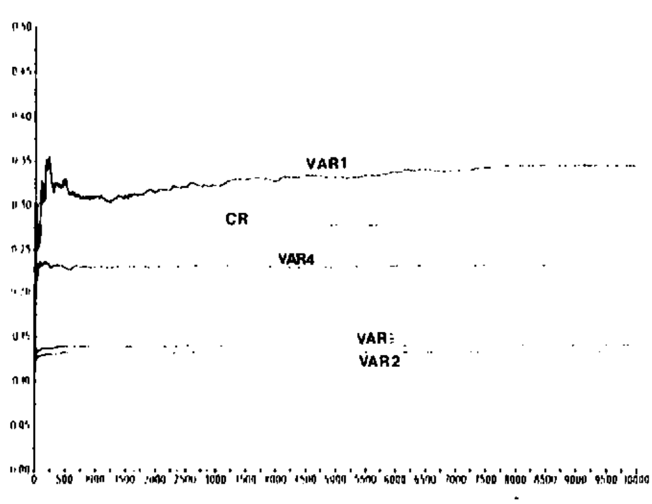

FIG. 1

Variances of $\widehat{var1}$ and $\widehat{var4}$:

It is well known ([5]) that bias($\widehat{var1}$)=0 and $var(\widehat{var1}) = \dfrac{2\, var^2(\hat{\theta}_q)}{L-1} \xrightarrow[L\to\infty]{} 0$ (30)

It is clear that bias($\widehat{var4}$) $\xrightarrow[K\to\infty]{} 0$ and assuming bias($\widehat{var4}$)=0 and $\widehat{var4}$ follows as $\hat{g}(\hat{\theta}_q)$ an $\alpha\chi^2_Q$ law

($Q \approx 2(N-P)$) one obtains: $var(\widehat{var4}) \xrightarrow[K\to\infty]{} \dfrac{2\, var^2(\hat{\theta}_q)}{LQ}$ (with $Q\approx 2(N-P)$ $\xrightarrow[L\to\infty]{} 0$ (31)

So $var(\widehat{var4})$ << $var(\widehat{var1})$ (we can observe it on fig.1)

Using (30) and (31) we can remark (P=1,N=16) that a precision of 10% over var($\hat{\theta}_q$) requires 200 samples for $\widehat{var1}$ and 20 for $\widehat{var4}$ (If bias($\widehat{var4}$)=0).

<u>3.6 Sensitivity to signal distribution:</u>

These previous simulations and calculations are available if the vectors used to compute $\hat{\Gamma}$ (periodogram) are independant and follow a gaussian law. A simulation was performed using vectors which were issued from a DFT of vectors following a uniform law: Conditions are P=1, N=16 (ULA), f=0.380, θ_1 = 5 degrees, K=100, S_1=0.1. 10000 independant measurings have been processed and the test of fit has been processed over 250 groups of 40 samples. Mean and variances of $\hat{\theta}_q$ are estimated over the 10000 samples.

Testing $\hat{\theta}_q$ gaussian distribution: Probability of false rejection: 5% , Observed rejection: 6.0%

Testing $\hat{g}(\hat{\theta}_q) \alpha \chi^2_Q$ distribution:Probability of false rejection: 5%, Observed rejection: 3.2% (Q=18.4)

$\widehat{var1}$=0.229 $\widehat{var2}$=0.137 $\widehat{var3}$=0.144 $\widehat{var4}$=0.225 CR bound=0.277

Exemple: case (P, S_1, K)=(1, 0.1, 50) evolution of the estimates versus the number of samples:

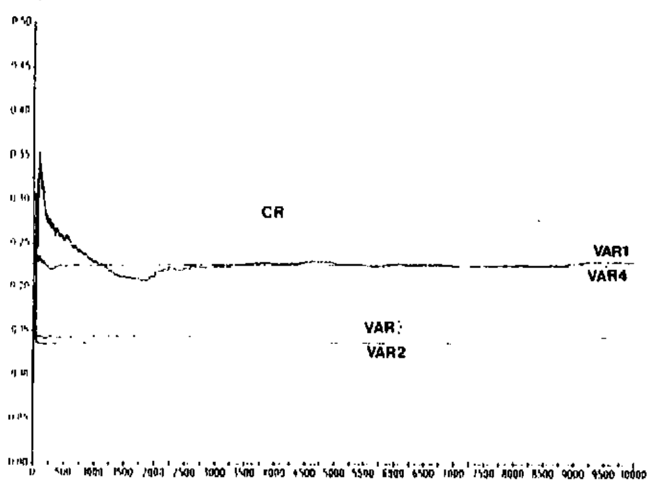

FIG. 2 K 50 P 1 teta1 5deg S1 0.1 N 16 non gaussian signal

$\widehat{var1}$ is quite sensible to (ie take into account) signal distribution, while $\widehat{var2}$, $\widehat{var3}$, $\widehat{var4}$ seem to be the same as in the gaussian case.

<u>3.7 Sensitivity to number of sources estimation:</u>

Previous simulations and calculations are only valuable if the number of sources is well estimated. To test the previous results in a case where the number of sources is overestimated, the following simulation was performed: P=1,S_1=0.1, θ_1= 5 deg., f=0.380, K=500, N=16,$\hat{P}$=3,10000 samples.

Testing $\hat{\theta}_q$ gaussian distribution: Probability of false rejection: 5% , Observed rejection: 4.8%

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Testing $\hat{g}(\hat{\theta}_q)\,\alpha\,\chi^2_Q$ distribution: Probability of false rejection: 5%, Observed rejection:3.2% (Q=23.1)

$\widehat{var1}$=0.0324 $\widehat{var2}$=0.0258 $\widehat{var3}$=0.0294 $\widehat{var4}$=0.0299 CR bound=0.0277

CONCLUSION

$\widehat{var4}$ is an estimator of var($\hat{\theta}_q$) which is asymptotically unbiaised and has **a smaller variance than the <u>quasi efficient</u> empirical variance estimator** $\widehat{var1}$ (*) so that one can use only a few measurements to compute it with good precision. It presents an other advantage: it does not need an estimate of the power as the estimator using the Cramer-Rao bound ($\widehat{var2}$) and *above all* it can be used even **in the general case of many sources**. $\widehat{var4}$ seems to be a better estimator than $\widehat{var3}$ which is also derivated from the expression of actual bearing precision but does not really exploit the relation or possible correlation between $\hat{\theta}_q$ and $\hat{g}(\hat{\theta}_q)$. Drawback is the relative insensibility to non gaussian signal when $\widehat{var1}$ shows that non gaussian hypothesis can obviously modify var($\hat{\theta}_q$).

(*) this is not a contradiction, because $\widehat{var4}$ uses an other variable than $\hat{\theta}_q$.

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