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ON THE RÔLE OF THE SECOND-ORDER PARABOLIC WAVE EQUATION IN NON-LINEAR ACOUSTICS

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Abstract

A short review of the second-order parabolic wave equation and its recently obtained solutions for monotonic and bifrequency finite-amplitude waves radiated by plane piston projectors in thermo-viscous fluids is provided herein.

1. Introduction

The purpose of this paper is to provide a very brief review of the role played by the second-order parabolic wave equation in nonlinear acoustics. With this in mind the discussion has been kept as free from symbolic content as possible in order to emphasize the significance of the principal analytical and numerical results that have thus far been obtained.

In order to derive the second-order parabolic wave equation, one begins by considering the general second-order wave equation for acoustical disturbances in an isotropic, homogeneous, thermo-viscous fluid which is given by including viscous and thermal losses in Westervelt's [1] lossless form of the equation as

$$\left\{(1-\Lambda\partial_t)\,\nabla^2 - c_o^{-2}\partial_t^2\right\} p' = (\beta/\rho_o c_o^4)\;\partial_t^2 p'^2 \tag{1}$$

Where p' is the excess (i.e. acoustic) pressure, ρ_o is the equilibrium density of the fluid, c_o is the small-signal-speed-of-sound, β is the coefficient of nonlinearity and Λ is the thermo-viscous damping coefficient.

Putting $t' = t - z/c_o$ and $z' = z$ gives $\partial_t = \partial_{t'}$ and $\partial_z = \partial_{z'} - c_o^{-1}\partial_{t'}$. Assuming further that the rate-of-change of the time waveform is much greater with respect to t' than z', which is a most reasonable hypothesis for progressive waves, then $\partial_{z'}^2 \ll \partial_{z't'}^2$ so that in coordinates $\sigma = \beta\varepsilon_o k_o z$ and $\tau = \omega_o t'$ scaled with respect to an arbitrary reference frequency ω_o, eq. (1) becomes

$$\partial_\tau\left\{\partial_\sigma - (1/\Gamma_o)\partial_\tau^{\,2} - P'\,\partial_z\right\}P' - (1/2\sigma_o)\,\nabla_\perp^2\,P = 0,\; P = p/p_o \tag{2}$$

where $\nabla^2 = \frac{L_y}{L_x}\partial_{x'}^2 + \frac{L_x}{L_y}\partial_{y'}^2$, $x' = x/L_x$, $y' = y/L_y$; L_x and L_y being characteristic lengths in the x and y directions. Likewise, p_o is an arbitrary reference pressure (e.g. the peak pressure at the source), $\sigma_o = \beta\varepsilon_o k_o r_o$, $\varepsilon_o = p_o/\rho_o c_o^2$, $r_o = A/\lambda_o$, $A = L_x L_y$, and $\Gamma_o = \beta\varepsilon_o k_o/\alpha_o$, with $\alpha_o = \Lambda\omega_o^2$.

Eq. (2) is the thermo-viscous form of Zabolotskaya and Khokhlov's [2] equation derived by Kuznetsov [3]. Exact but rather intractable lossless solutions of this equations have been deduced by Rudenko [4] and by Vinogradov and Vorb'ev [5]. In the frequency domain eq. (2) becomes

$$\left\{\partial_\sigma + (\Omega^2/\Gamma_o) - (i/2\sigma_o)\nabla^2\right\}P_\Omega = (i\Omega/2)\ F_\Omega(P^2),\ \Omega = \omega/\omega_o \qquad (3)$$

where $F_\Omega(\ ) = \int_{-\infty}^{\infty}(\ )e^{i\Omega\tau}\,d\tau$ is the Fourier Transform of the function in parenthesis.

Eq. (3) is the second-order parabolic wave equation with which this paper is concerned. Before considering its properties however, we note at this point that as $\sigma_o \to \infty$, eq. (2) assumes the plane wave form of Burgers' equation, and likewise eq. (3) reduces to the form of the latter in the frequency domain. Hence σ_o is a natural perturbation parameter which has been used by Zabolotskaya and Khokhlov [6] and by Vorob'ev and Slavin [7] to obtain lossless asymptotic solutions of eq. (2) and hence of eq. (3). A considerable research effort is required however to deduce tractable solutions of eq. (3) for practical acoustic problems.

2. Weak Finite-Amplitude Waves

2.1 Monotonic Excitation

For the case of weak finite-amplitude waves, eq. (3) can be solved by the method of successive approximations. Using this approach, Rudenko, Soluyan, and Khokhlov [8] obtained a solution for the second harmonic field formed in a lossless fluid (i.e. $\Gamma_o = \infty$) via nonlinear self-interaction of a weak monotonic finite-amplitude wave of frequency ω_o radiated by an axisymmetrically excited circular piston. A generalization of this solution derived by Fenlon and Kesner [9] for second harmonic generation in a thermo-viscous fluid is given along the beam axis with $R = \sigma/\sigma_o$ and $a_o = \sigma_o/\Gamma_o$ as

$$P_{2\Omega}(R) = \frac{\sigma_o/2}{1-iR}\left\{E_1[-i2a_o] - E_1[-i2a_o\ (1-iR)]\right\} \times \ \exp[-4a_oR - i2a_o] \qquad (4)$$

$$\to \frac{\sigma_o/2}{1-iR}\ \ln(1-iR),\ \text{in a lossless fluid (i.e. } a_o = 0\text{) as given by Rudenko [8]} \qquad (4a)$$

$$\to (\Gamma_o/4)\left(e^{-2a_oR} - e^{-4a_oR}\right),\ \text{for } R \ll 1,\ a_o \gg 1 \qquad (4b)$$

where $E_1(x) = \int_x^\infty \frac{e^{-x'}}{x'}\,dx'$ is the exponential integral [10].

It should be noted that although eq. (4b) is identical to the plane wave solution of Burgers' equation derived by Naugol'nykh, Soluyan, and Khokhlov [11], in the present context it is simply a near-field, high-frequency approximation of eq. (4).

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For R << 1, eqs. (4a) and (4b) both give $P_{2\Omega} \to \sigma/2$, showing that the plan wave form of Burgers' equation holds, as expected, at short ranges within the near-field of a bounded aperture projector. On the other hand, the far-field solutions of eq. (4) is given by

$$P_{2\Omega}(R) \to (\Gamma_o/4) \left\{ i2a_o E_i[i2a_o] \frac{e^{-i2a_o - 4a_o R}}{R} - \frac{e^{-2a_o R}}{R^2} \right\}, \ R >> 1 \qquad (5a)$$

$$\to (\Gamma_o/4) \left(\frac{e^{-4a_o R}}{R} - \frac{e^{-2a_o R}}{R^2} \right), \ a_o >> 1 \qquad (5b)$$

For $a_o >> 1$, eq. (5a) shows that for significant regions of the far-field $P_{2\Omega} \to (\Gamma_o/4) \frac{e^{-2a_o R}}{R^2}$, and likewise, depending on the magnitude of a_o, eq. (5b) gives the same result, in agreement with an asymptotic solution of the spherical wave form of Burgers' equation derived by Webster and Blackstock [12].

Off axis, Rudenko, Soluyan, and Khokhlov [8] have shown that the second harmonic field is given by the square of the directivity function of the fundamental, in keeping with Lockwood, Muir, and Blackstock's [13] results.

2.2 Bifrequency Excitation

The difference-frequency signal formed in a thermo-viscous fluid via non-linear interaction of weak finite-amplitude primary waves of frequencies ω_1 and ω_2, simultaneously radiated by a plane piston projector has also been explicitly defined via solutions of eq. (3). For example, a far-field analysis by Fenlon [14] for the case of an arbitrarily excited bifrequency radiator subsequently led to a high frequency near-field solution by Novikov, Rudenko, and Soluyan [15] for the case of an axisymmetrically excited bifrequency radiator with Gaussian beam profiles at the face of the projector. As emphasized by Hobaek [16], the importance of the latter solution is that it gave explicit form to the difference-frequency field due to the fact that the Gaussian beam approximation enabled the field integrals to to integrated analytically. Novikov, Rybachek and Timoshenko [17] next generalized the previous analysis [15] to include diffraction effects due to the projector aperture, thus extending the solution to include both the near and far-field regions of the nonlinear interaction zone. A more general analysis for the case of an arbitrarily excited piston projector was carried out independently by Fenlon [18] and subsequently elaborated by Fenlon and McKendree [19]. In this analysis, the primary waves were expressed in Gauss-Laguerre modes of the linearized form of eq. (3) (derived by Kogelnik [20]), thus permitting the difference-frequency field formed by the primary wave fundamental (i.e. Gaussian) spacial modes to be recovered as a special case. Fenlon [21], [22] also obtained solutions of eq. (3) for the on-axis component of the difference-frequency field formed via nonlinearly interacting primary waves radiated by square and rectangular faced projectors, but only for the case of Gaussian primary modes. In this instance, for the case of a square-faced projector the axial difference-frequency field can be expressed as,

$$P_{\Omega-}(R) = \frac{(\kappa_-^2\sigma_o/2)}{1 - i\kappa_- R} \left\{ E_1 \; [-i(a_T\kappa_-/4)(1-i4R/\kappa_-)/(1-i\kappa_- R)] - E_1 \; [-i(a_T\kappa_-/4)(1+4R^2)/(1-i\kappa_- R] \right\} \times \exp \; [-a_{\omega_-} R - i(a_T\kappa_-/4)(1-i4/\kappa_-)/(i-i\kappa_- R)] \qquad (6)$$

$$\rightarrow \frac{(\kappa_-^2\sigma_o/2)}{1-i\kappa_- R} \; \ln\left(\frac{1+4R^2}{1-i4R/\kappa_-}\right), \text{ in a lossless fluid} \qquad (6a)$$

$$\rightarrow (\kappa_-^2 \; \sigma_o/2) \left\{ E_1[-i(a \; \kappa_-/4) \; (1-i4R/\kappa_-)] - E_1[-i(a \; \kappa_-/4] \right\} \times \exp \; [-a_{\omega_-} R - i(a \; \kappa_-/4)(1-i4R/\kappa_-)], \text{ for } R^2 \ll 1 \text{ in the near field} \qquad (6b)$$

where $\sigma_o = \beta\varepsilon_o k_o r_o$, $\varepsilon_o = (p_{01} + p_{02})/\rho_o c_o^2$, $k_o = (k_1+k_2)/2$, and $a_T = 2\sigma_o/\Gamma_o$ for $\omega_1 \simeq \omega_2$ with $\kappa_- = \omega_-/\omega_o$. Eq. (6) has also been shown [21], [22] to provide very good agreement with experimental results. Another interesting result is the so called 'gain' of a parametric array relative to that of the ideal 'virtual-end-fine-array' envisaged by Westrvelt [23] which is defined as $G = P_{\Omega_-}(R)/P^W_{\Omega_-}(R)$ where $P^W_{\Omega_-}(R) = (\kappa_-^2 \; \sigma_o/2a_T R) \; \exp(-a_{\omega_-} R)$. Since the value of G becomes independent of R in the far-field of the parametric array it is instructive to plot G/G_∞ as a function of R where G_∞ denotes the far-field gain. This function is depicted in Fig. 1 for $\omega_o/\omega_- = 1/\Omega_- = 5$, as a_T varies from 10^{-6} to 20.

Physically speaking, Fig. 1 shows that a parametric array only behaves as an ideal 'virtual-end-fire-array' throughout most of the nonlinear interaction zone for $a_T > 10$. Figure 1 also shows that the effective length of a parametric array can be considerable for small values of a_T. For example, if $a_T = 10^{-3}$, the effective parametric array length is greater than a thousand times the Rayleigh distance at the mean carrier frequency.

The directional characteristics of axisymmetric parametric arrays have also been evaluated by Fenlon and McKendree [19]. In Fig. 2, for example, the half-power beamwidth of a parametric array normalized with respedt to that of an ideal 'virtual-end-fire-array' is shown as a function of R for $a_T > 1$ and $\omega_o/\omega_- = 1/\Omega_- = 5$. The distinctive minima depected in these characteristics, which disappear for $a_T/\kappa_- < 1$, occur within the same region of R as the local maxima depicted in Fig. 1 Such features, which have been observed experimentally by Hobaek [16] are due to rapid phase changes within the near field of high frequency parametric arrays.

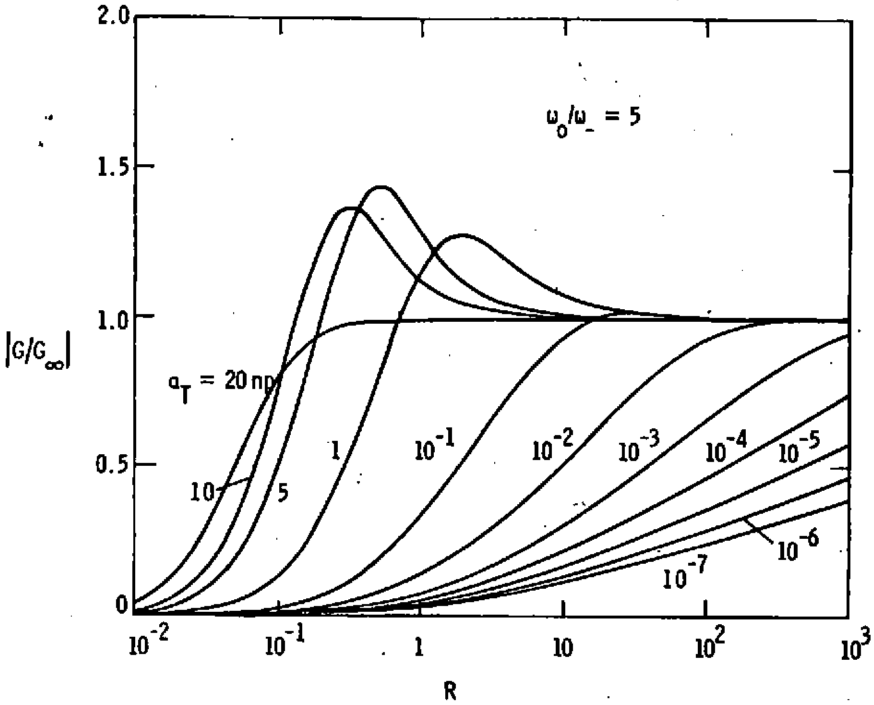

Fig. 1 - Parametric Array Gain Characteristics

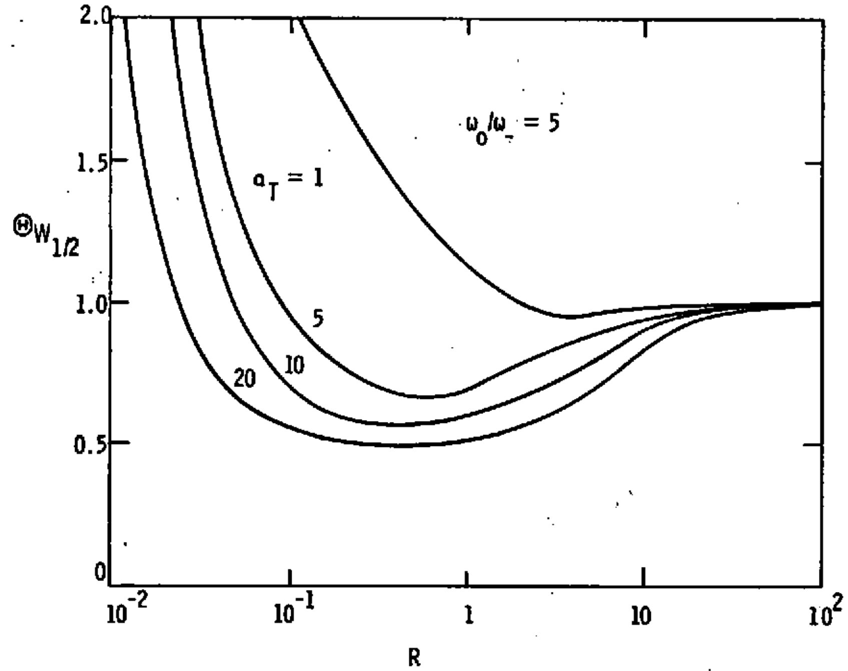

Fig. 2 - Parametric Array Beamwidth Characteristics

3. Strong Finite-Amplitude Waves

Significant waveform distortion and shock formation of initially monotonic waves radiated by an axisymmetrically excited piston operating at strong finite-amplitudes in a thermo-viscous fluid have recently been investigated by Bakhvalov, Zhileikin, Zabolotskaya, and Khokhlov [24], and by McKendree and Fenlon [25], via numerical analysis of eq. (3). Typical results obtained for the case of Gaussian mode excitation are shown in Fig. 3, the on-axis time waveform being depicted at ranges where the maximum nonlinear distortion has occurred.

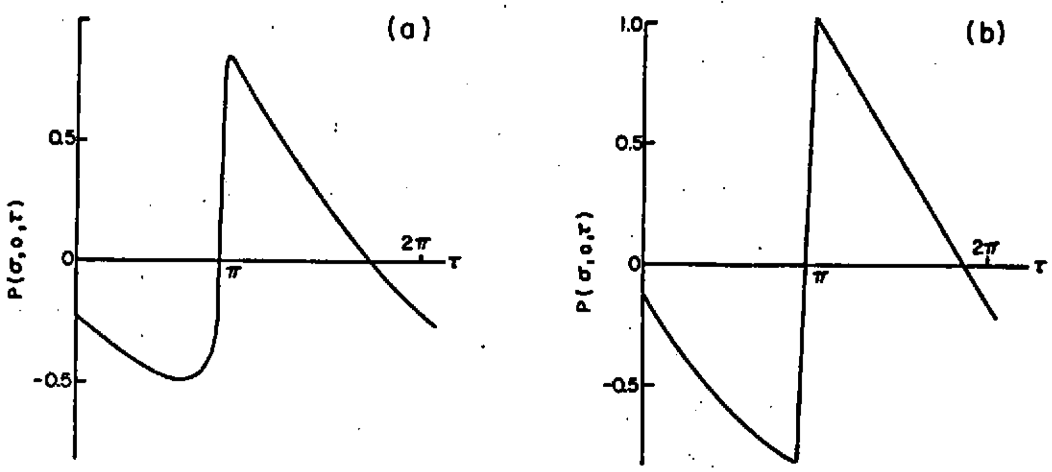

Figure 3. On-axis time waveforms

(a) $\sigma = 2$, $\sigma_o = 3.25$, $\Gamma_o = 65$; (b) $\sigma = 1.8$, $\sigma_o = 10$, $\Gamma_o = 200$

Unlike the shock wave solutions of Burgers' equation, these waveforms clearly exhibit 'd.c. bias' resulting from phase shifts introduced via diffraction effects. These results are in keeping with experiment. McKendree and Fenlon [25] have also computed axisymmetric difference-frequency fields formed by strong finite-amplitude primary waves, but much work remains to be done before a complete summary of these results is available.

4. Conclusions

In this brief summary, only the primary references to recent investigations of the second-order parabolic wave equation have been cited. It is hoped that the reader will at least have obtained an overview of this rapidly expanding field of investigation.

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