# NONLINEAR VIBRATION ANALYSIS OF COMPOSITE LAMINATED PLATES USING THE FEM

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The present study is concerned with the nonlinear vibration characteristics of laminated composite plates. Considering the geometric nonlinearity of plates, the vonKarman large amplitude theory is used, and the finite element method (FEM) is proposed for the present nonlinear model, using Hamilton's principle, the equation of motion of the composite laminated plates is established. Comparisons of the present results with those published in open literatures show the accuracy and correctness of the present methodology. Moreover, the nonlinear vibration of composite laminated plates under the harmonic excitation force is discussed in time domain.

Keywords: composite laminated plates, nonlinear vibration, finite element method

## 1. Introduction

The vibration of plates results from the internal interaction of elastic force and inertial force. Because the composite laminated plates have high stiffness, strength and low weight, they are often used in commercial and military aircraft, and the vibration of the composite laminated plates is concerned by many researchers.

Numerous investigators have studied the linear vibration of laminated composite plates [1-3]. But recently, the environment situation of the plates is always in supersonic speed and extreme high heat, so the influences of nonlinearity must be considered in the research. Houmat [4] presented solutions to the geometrically nonlinear flexural free vibration of a rectangular composite plate composed of variably spaced rectilinear and parallel fibers. Oh et al. [5] analyzed the nonlinear transient response of stiffened composite plates with thermal loads. Amabili and Farhadi [6] studied the nonlinear forced vibrations of isotropic and laminated composite rectangular plates and compared with the classical and shear deformable theories. Saha et al. [7] researched the large amplitude free vibration of a thin square plate with different non-classical boundary conditions.

In the present study, the nonlinear vibration characteristics of laminated composite plates with the geometric large deformation are investigates, the harmonic excitation force is applied at the central of the plates, and the boundary conditions of the plates are clamped at all edges. The effects of the nonlinearity are discussed in time domain.

## 2. Equation of motion of the composite laminated plates

The composite laminated plate is shown in Fig.1. Three variables u, v and w are used to denote the displacements of the plate in the x, y and z directions. The coordinate origin is chosen such that the lower left corner is on the middle surface.

The displacement fields of the plates are given as

$$u = u_0 - z\frac{\partial w_0}{\partial x}, v = v_0 - z\frac{\partial w_0}{\partial y}, w = w_0 \tag{1}$$

where z is the transverse coordinate of the panel, and $u_0$, $v_0$ and $w_0$ are the mid-plane displacements.

According to the von-Karman large deformation theory, the strain–displacement relations can be expressed as

$$\varepsilon=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial y}\\ \dfrac{\partial v_0}{\partial x}+\dfrac{\partial u_0}{\partial y}\end{bmatrix}+\begin{bmatrix}-z\dfrac{\partial^2 w_0}{\partial x^2}\\ -z\dfrac{\partial^2 w_0}{\partial y^2}\\ -2z\dfrac{\partial^2 w_0}{\partial x\partial y}\end{bmatrix}+\begin{bmatrix}\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2\\ \dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial y}\right)^2\\ \dfrac{\partial w_0}{\partial x}\dfrac{\partial w_0}{\partial y}\end{bmatrix}=\left[\varepsilon_0\right]+z\left[k\right]+\left[\varepsilon_l\right], \tag{2}$$

The constitutive equation of the *k*th lamina is

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{33}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\left[Q\right]\left\{\varepsilon_c'\right\},$$

where $Q_{11}=\dfrac{E_1}{1-\mu_1\mu_2}$, $Q_{12}=Q_{21}=\dfrac{E_1\mu_2}{1-\mu_1\mu_2}=\dfrac{E_2\mu_1}{1-\mu_1\mu_2}$, $Q_{22}=\dfrac{E_2}{1-\mu_1\mu_2}$, and $Q_{33}=G$.

The constitutive equation of the *k*th lamina is transformed into the global coordinate system

$$\left\{\sigma\right\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\left[D\right]\left[Q\right]\left[D\right]^T\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\left[\bar{Q}\right]\left\{\varepsilon_c\right\}, \tag{3}$$

Because the triangular plate element can adapt multiple boundary shapes, this element is used to discrete the mid-plane and the displacements of the node can be divided into the bending displacements (w, $\varphi_x, \varphi_y$ )and the axial displacements (u, v). So the displacements of the element' node can be expressed as follows

$$w_0=\left[H_w\right]\left[w_b\right], u_0=\left[H_u\right]\left[w_u\right], v_0=\left[H_v\right]\left[w_v\right], \tag{4}$$

where

$$\left\{w_b\right\}=\left\{w_1\quad \phi_{x1}\quad \phi_{y1}\quad w_2\quad \phi_{x2}\quad \phi_{y2}\quad w_3\quad \phi_{x3}\quad \phi_{y3}\right\}^T,$$

$$\left\{w_u\right\}=\left\{u_1\quad u_2\quad u_3\right\}^T, \left\{w_v\right\}=\left\{v_1\quad v_2\quad v_3\right\}^T,$$

$$\left\{H_w\right\}=\left\{N_1\quad N_{x1}\quad N_{y1}\quad N_2\quad N_{x2}\quad N_{y2}\quad N_3\quad N_{x3}\quad N_{y3}\right\}^T,$$

$$\left\{H_u\right\}=\left\{L_1\quad L_2\quad L_3\right\}^T, \left\{H_v\right\}=\left\{L_1\quad L_2\quad L_3\right\}^T,$$

$$N_i=L_i+L_i^2L_j+L_i^2L_m-L_iL_j^2-L_iL_m^2,$$

$$N_{xi}=b_jL_i^2L_m-b_mL_i^2L_j+\left(b_j-b_m\right)L_iL_jL_m/2,$$

$$N_{yi}=c_jL_i^2L_m-c_mL_i^2L_j+\left(c_j-c_m\right)L_iL_jL_m/2,$$

$$a_i=x_jy_m-x_my_j, b_i=y_j-y_m, c_i=x_m-x_j,$$

$$i\text{=1-2-3}, j\text{=2-3-1}, m\text{=3-1-2},$$

in which $L_1, L_2, L_3$ are area coordinates of the triangular plate element.

Hamilton's principle is used to formulate the governing equation of motion, and its expression is given as [8]:

$$\delta\int_{t_1}^{t_2}\left(T-U\right)dt+\delta\int_{t_1}^{t_2}W\,dt=0 \tag{5}$$

According to Eqs. (1)-(5), the equation of motion of the element is obtained, and assembling the element matrices into the global ones, the equation of motion of the whole composite laminated plates can be obtained as

$$[M]\{\ddot{w}\}+[C]\{\dot{w}\}+([K_1]+[K_2])\{w\}=\{F\} \tag{6}$$

where $w=[u_1,v_1,w_1,\varphi_{x1},\varphi_{y1}\cdots u_n,v_n,w_n,\varphi_{n1},\varphi_{n1}]^T$ in which n is the number of the degrees of freedom is the nodal displacement vector, $[M]$ is the mass matrix, $[C]$ is the structural damping, $[K_1]$ and $[K_2]$ are the linear and nonlinear stiffness matrices, $\{F\}$ is the excitation force vector.

In order to calculate the static response of the plate, the tangent stiffness matrix is given as follows

$$K_T=\left(\begin{bmatrix} 0_{6*6} & 0_{6*9} \\ 0_{6*6} & \int_v [D_{l2}]^T \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}[D_{l2}]dv \end{bmatrix}+\left([K_1^e]+[K_2^e]\right)\right).$$

## 3. Numerical simulations and discussions

### 3.1 Validations of the formulations and codes

The numerical simulations are performed by the MATLAB software. In order to verify the correctness of the governing equation of motion and the MATLAB programs, the static responses of the anisotropy plates are compared with those of Reddy [9] as shown in Table 1. The material properties of the composite panel used in the calculation are $E_1$ = 3×10$^6$Pa, $E_2$ = 1.28×10$^6$Pa, $G_{12}$ = 0.37×10$^6$Pa, $\rho$ = 1600kg/m$^3$, $\mu_{12}$ = 0.32, a=b=12(in), and h=0.138(in). The boundary conditions considered here are

(a) Simply supported case

$$u_0=v_0=w_0=0 \text{ at x = 0, a and y = 0, b.}$$

(b) All edges clamped

$$u_0=v_0=w_0=\varphi_x=\varphi_y=0\,.$$

Table 1 The static response of all edges clamped anisotropy plates under uniform load.

| | J.N.Reddy [9] | present |
|---|---|---|
| $q_0$/psi | $w_0$/in | $w_0$/in |
| 0.5 | 0.0294 | 0.0326 |
| 1.0 | 0.0552 | 0.0592 |
| 4.0 | 0.1456 | 0.1424 |
| 8.0 | 0.2054 | 0.1958 |
| 12.0 | 0.2450 | 0.2313 |
| 16.0 | 0.2754 | 0.2587 |
| 20.0 | 0.3006 | 0.2816 |

Table 2 Comparison of nonlinear frequency ratios (ωNL/ωL) of simply supported anisotropy plates

| w/h | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|
| Perturbation method[10] | 1.0196 | 1.0761 | 1.1642 | 1.2774 | 1.4097 |
| Finite element[11] | 1.0134 | 1.0518 | 1.1154 | 1.1946 | 1.2967 |

| | | | | | |
|---|---|---|---|---|---|
| FE time history[12] | 1.0190 | 1.0739 | 1.1597 | 1.2699 | 1.3987 |
| present | 1.0137 | 1.054 | 1.1182 | 1.2034 | 1.3063 |

Further, the variation of nonlinear frequency ratio ($\omega_{NL}/\omega_L$, where subscripts NL and L represent the nonlinear and linear cases) with respect to the maximum amplitude ($w_{\max}/h$, where $w_{\max}$ is the maximum amplitude of the plate) is evaluated for simply supported square plates, and it is compared with those in the open literatures are shown as in Tables 2. The material properties of the composite panel used in the calculation are: $E_1 = 4\times10^6$, $E_2 = 1\times10^6$, $G_{12} = 0.5\times10^5$, $\rho = 1$, $\mu_{12} = 0.32$, a=b=1000×h, and $h=1\times10^{-3}$,

It is observed from Tables 1 and Tables 2 that the present results are in good agreement with those in the open literatures, which verifies that the governing equation obtained in this paper and the MATLAB programs are correct.

### 3.2 Nonlinear vibration analysis

The material properties used in this section are: $E_1 = 40\times10^9$Pa, $E_2 = 1\times10^9$Pa, $G_{12} = 0.5\times10^9$Pa, $\rho = 1000$kg/m$^3$, $\mu_{12} = 0.25$, a=b=0.3(m), and single lamina is $h=1.5\times10^{-3}$(m). The ply angle of the two layer is $[90^\circ/0^\circ]$. The boundary conditions for the simply supported plate with immovable edges are as follows:

$$u_0 = v_0 = w_0 = \varphi_x = 0 \text{ at x=0, a and } u_0 = v_0 = w_0 = \varphi_y = 0 \text{ at y=0, b.}$$

The first natural frequency for linear vibration of the plate is calculated as $\omega_L$ = 100.4 Hz. A harmonic force is applied at the center of the plate. It is assumed that the damping matrix is eye(n) ×0.00525 (n is the number of the degrees of freedom ), the frequency ratio of the force is $\omega_{NL}/\omega_L$, and the maximum force is f. The transverse displacement w of the center point is researched.

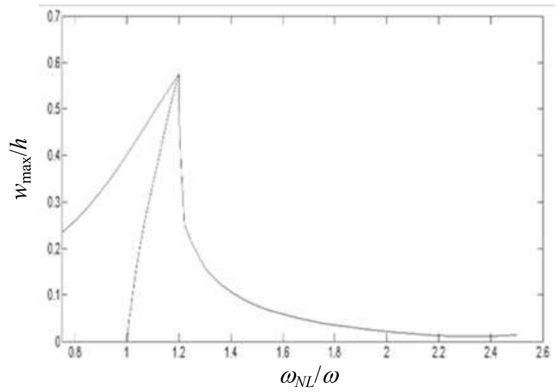


Fig. 1 Frequency-response behavior curve for f=10N

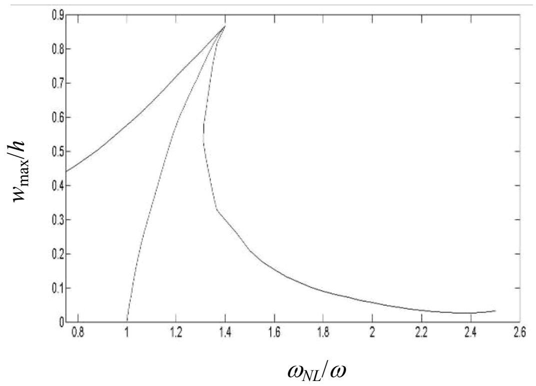


Fig. 2 Frequency-response behavior curve for f=20N

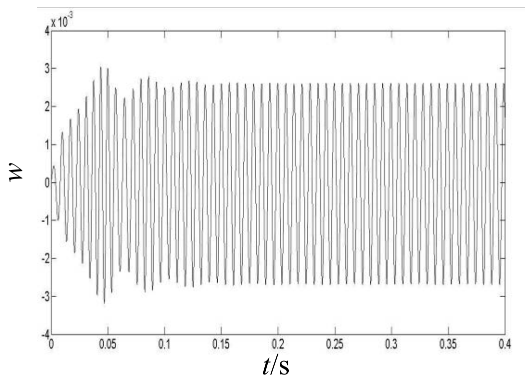


Fig. 3 Vibration time history of the plate

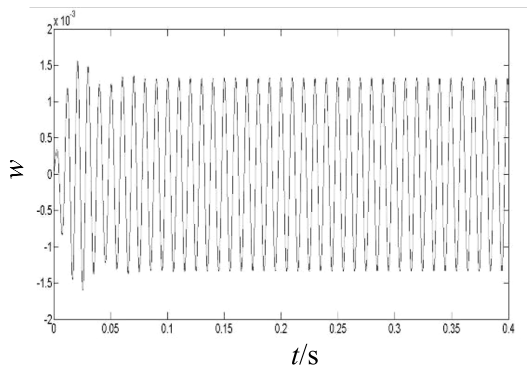


Fig. 4 Vibration time history of the plate

for $\omega_{NL}/\omega_L = 1$ and f = 10N. for $\omega_{NL}/\omega_L = 1$ and f = 10N.

Initially, the maximum force $f = 10N$ is considered, the frequency response curve of the plate under harmonic force is shown in Fig. 1. It can be seen from the figure that because of the nonlinear factor the hard spring behavior is observed. For $f = 20N$, the similar phenomenon can be obtained as shown in Fig. 2.

Next, when the force is $f = 10N$, and the frequency ratio is $\omega_{NL}/\omega_L = 1$, the vibration time history of plate is presented in Fig. 3. From Fig. 2, it can be seen that when the frequency ratio is $\omega_{NL}/\omega_L = 1.4$, the maximum amplitude $w_{\max}$ reaches the largest value, and for this case the vibration time history of the plate is shown in Fig. 4.

## 4. Conclusion

In this paper, the nonlinear vibration characteristic of laminated composite plates is investigated, and the geometric large deformation is considered. The classical von-Karman large deformation theory and the finite element method are used. From the numerical results, the following conclusions can be drawn:

(1) The nonlinearity should be considered to analyze the vibration of the plate with large deformation, consequently, the rigidity of the plate is increased, and the frequency response curve may shift toward right, which reflects hard spring behavior of the system.

(2) With the excitation force increasing, the shifting angle of the frequency response curve is also increased, and the increasing trend of the maximum amplitude is reduced.

(3) For the different excitation forces, the maximum amplitudes are all on the backbone curve.

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