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THE REFRACTION OF ULTRASOUND IN GRAINY AND INHOMOGENEOUS FIBRE REINFORCED MATERIALS

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1. Introduction

The necessity of non-destructively inspecting cast steels, weldments, fibre reinforced materials, and other inherently anisotropic materials has stimulated renewed interest in ultrasonic wave propagation in anisotropic and inhomogeneous media. In a recent article, Abrahams & Wickham (1991a), the authors considered the refraction of a horizontally polarized shear source at the "fusion interface" between an isotropic material $\mathcal{F}$ and an inhomogeneous transversely isotropic solid $\mathcal{W}$. The inhomogeneity in the latter was chosen to be characteristic of the type of crystalline structures produced in austenitic steel welds. In these it is found that, as the weld metal cools, the crystals align themselves so that an axis of symmetry lies parallel to the direction of heat flow. This process of "epitaxial growth" produces a coherent macroscopic structure characterized by a "grain angle" whose principal variation is with distance from the fusion interface. The authors modelled the latter by allowing the direction θ of the zonal axis or axis of symmetry of the crystals to be a function of the perpendicular distance x from the interface, see figure 1. The mathematical boundary value problem was solved exactly, and, in the high frequency limit, a uniform asymptotic expansion for the displacement vector was found. It was shown that in this limit, and for a wide range of material constants, the refracted energy could experience total internal reflection in the anisotropic domain. In this paper we outline the analysis for the much more complicated case where the displacement vector lies in the plane of propagation (two-dimensional plane strain). We evaluate the field analytically and demonstrate a relatively simple qualitative determination of the structure of the refracted waves. As in the SH-case we show that total internal reflection can occur for certain grain structures and elastic moduli. Our conclusions appear to be highly significant in the design of inspection procedures for structurally important welds.

2. Formulation of the refraction problem for a compressional source

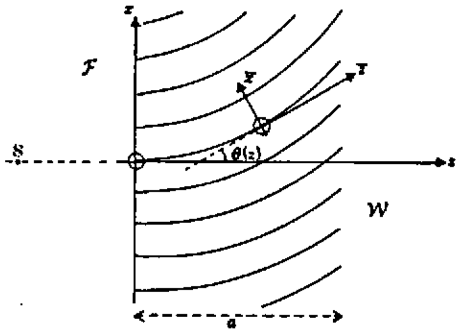
Consider a set of *local* cartesian coordinates $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{y} = y$ and $\bar{z}$ parallel to the zonal axis at that point, see figure 1. Relative to such a frame of reference, Hooke's law may be written in the form

$$\bar{\tau}_{ij} = \bar{c}_{ijkl}\bar{\epsilon}_{kl} \qquad (1)$$

where the fourth order tensor $\bar{c}_{ijkl}$ contains five independent entries. We shall assume that the density of both materials are constant and $\bar{c}_{ijkl}$ is a constant function of position in $\mathcal{W}$, i.e. all the anisotropic crystals are elastically identical. In this case it is evident that the general stress-strain law relative to cartesian coordinates (x, y, z) is of the form

$$\tau_{ij} = c_{ijkl}(\theta)\epsilon_{kl}. \qquad (2)$$

Figure 1: The geometry of the problem showing the source position S, the grain angle $\theta(z)$ and the grain coordinates $(\bar{x}, \bar{z})$



The orthogonal transformation describing a rotation θ about the y-axis is given by the second order tensor

$$a_{ij}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3}$$

so that

$$\bar{c}_{ijkl} = a_{ip}(\theta)a_{jq}(\theta)a_{kr}(\theta)a_{ls}(\theta)c_{pqrs}(\theta), \tag{4}$$

or

$$c_{pqrs} = a_{pi}(\theta)a_{qj}(\theta)a_{rk}(\theta)a_{sl}(\theta)\bar{c}_{ijkl}. \tag{5}$$

This last result conveniently expresses the Hooke's tensor in terms of the orientation of the zonal axis and hence as a function of z. For completeness we display the matrix form of equation (1) using the engineering notation for strain, namely

$$\begin{pmatrix} \bar{\tau}_{11} \\ \bar{\tau}_{22} \\ \bar{\tau}_{33} \\ \bar{\tau}_{23} \\ \bar{\tau}_{13} \\ \bar{\tau}_{12} \end{pmatrix} = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & 0 \\ \bar{c}_{12} & \bar{c}_{11} & \bar{c}_{13} & 0 & 0 & 0 \\ \bar{c}_{13} & \bar{c}_{13} & \bar{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{c}_{66} \end{pmatrix} \begin{pmatrix} \bar{e}_{11} \\ \bar{e}_{22} \\ \bar{e}_{33} \\ \bar{e}_{23} \\ \bar{e}_{13} \\ \bar{e}_{12} \end{pmatrix} \tag{6}$$

for a hexagonal material with

$$\bar{c}_{66} = (\bar{c}_{11} - \bar{c}_{12})/2 \tag{7}$$

and

$$\bar{e}_{ij} = \bar{\epsilon}_{ij}, \quad i = j, \qquad \bar{e}_{ij} = 2\bar{\epsilon}_{ij}, \quad i \neq j. \tag{8}$$

We now consider a state of plane strain in which $\mathbf{u} = (u(x,z), 0, w(x,z))$ so that the only non-vanishing strain components are $\epsilon_{xz}, \epsilon_{xx}$ and ϵ_{zz} in $\mathcal{F} \cup \mathcal{W}$. It is supposed that an isotropic time-harmonic line source is situated at $(0, z')$, where $z' < 0$ and is reflected and refracted at the plane interface $x = 0$ between $\mathcal{F}$ and $\mathcal{W}$. Thus, if $\phi(x,z)$, $\psi(x,z)$ are the usual Lamé potentials for the motion in $\mathcal{F}$, we have

$$u_i = \frac{\partial\phi}{\partial x_i} + \epsilon_{i2j}\frac{\partial\psi}{\partial x_j}, \tag{9}$$

where, for a compressional source of strength ϕ_0,

$$\left.\begin{array}{rcl}(\nabla^2+k^2)\phi &=& \phi_0\delta(x)\delta(z-z') \\ (\nabla^2+K^2)\psi &=& 0\end{array}\right\},\ (x,z)\in\mathcal{F}, \tag{10}$$

in which

$$k^2=\rho\omega^2a^2/(\lambda+2\mu),\quad K^2=\rho\omega^2a^2/\mu \tag{11}$$

are the dimensionless compression and shear wavenumbers respectively, λ, μ are the Lamé constants for $\mathcal{F}$ and a is a typical length scale of the grain structure in $\mathcal{W}$. In (10) and all the subsequent theory, all lengths have been scaled on a. In the inhomogeneous material $\mathcal{W}$, the general equation of motion is

$$\frac{\partial}{\partial x_j}\bar{c}_{ijkl}(\theta)\frac{\partial u_k}{\partial x_l}+N^2u_i=0,\ (x,z)\in\mathcal{W} \tag{12}$$

where

$$\bar{c}_{ijkl}(\theta)=c_{ijkl}(\theta)/\bar{c}_{44} \tag{13}$$

and $N^2=\rho\omega^2a^2/\bar{c}_{44}$. Our task is to determine a solution of (12) with (10) which satisfies the welded interface boundary conditions

$$\mathbf{u}(x,0^+)=\mathbf{u}(x,0^-),\ -\infty<x<\infty \tag{14}$$

and

$$\left.\begin{array}{rcl}\tau_{xz}(x,0^+) &=& \tau_{xz}(x,0^-) \\ \tau_{zz}(x,0^+) &=& \tau_{zz}(x,0^-)\end{array}\right\},\ -\infty<x<\infty \tag{15}$$

where τ_{ij} is the stress tensor, and represents outgoing waves at infinity. In $z<0$ we have the stress-displacement relations in the form

$$\left.\begin{array}{rcl}\mu^{-1}\tau_{xz} &=& 2\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x}\right)+K^2\psi \\ -\mu^{-1}\tau_{zz} &=& 2\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z}\right)+K^2\phi\end{array}\right\} \tag{16}$$

while in $z>0$

$$\bar{c}_{44}^{-1}\tau_{i3}=\bar{c}_{i3kl}(\theta)\frac{\partial u_k}{\partial x_l}. \tag{17}$$

3. Formal solution of the transmission problem

We take Fourier transforms of the system of partial differential equations and boundary conditions with respect to the spatial variable x. In particular, if

$$\mathbf{U}(\alpha,z)=\int_{-\infty}^{\infty}\mathbf{u}(x,z)e^{i\alpha Nx}\,dx, \tag{18}$$

we find that $\mathbf{U}(\alpha,z)$ satisfies the ordinary differential equation

$$\frac{d}{dz}\mathbf{P}\frac{d\mathbf{U}}{dz}-iN\alpha\left\{(\mathbf{Q}+\mathbf{Q}^{\mathbf{T}})\frac{d\mathbf{U}}{dz}+\frac{d\mathbf{Q}}{dz}\mathbf{U}\right\}+N^2(\mathbf{I}-\alpha^2\mathbf{R})\mathbf{U}=0,\ z>0, \tag{19}$$

where

$$P_{il}(z) = \tilde{c}_{i33l}(\theta(z)), \quad Q_{il}(z) = \tilde{c}_{i31l}(\theta(z)), \quad R_{il}(z) = \tilde{c}_{i11l}(\theta(z)). \tag{20}$$

Assuming that we can find precisely two linearly independent solutions of this equation which are source-free in $z > 0$, we write the general solution for the motion in $\mathcal{W}$ as

$$\mathbf{U}(\alpha, z; z') = U^{qL}(\alpha, z')\mathbf{S}^{qL}(\alpha, z) + U^{qT}(\alpha, z')\mathbf{S}^{qT}(\alpha, z). \tag{21}$$

The notation here is intended to echo the fact that, at any particular point, we expect that the displacement field is representable as the superposition of a "quasi-longitudinal" and a "quasi-transverse" wave. For the moment we proceed assuming these special functions are known and that all we need to complete the solution is to calculate the undetermined coefficients U^{qL} and U^{qT}. Returning to equations (10), we find that the appropriate solutions for $(\Phi(\alpha, z; z'), \Psi(\alpha, z; z'))$, the Fourier transforms of $\phi(x, z; z')$ and $\psi(x, z; z')$, are of the form

$$\Phi(\alpha, z; z') = \Phi^P(\alpha, z')e^{Nz\gamma(\alpha)} - \frac{\phi_0}{2N\gamma(\alpha)}e^{-N\gamma(\alpha)|z-z'|}, \; z < 0 \tag{22}$$

$$\Psi(\alpha, z; z') = \Psi^S(\alpha, z')e^{Nz\delta(\alpha)}, \; z < 0, \tag{23}$$

where

$$\left.\begin{aligned} \gamma(\alpha) &= (\alpha^2 - k^2/N^2)^{\frac{1}{2}}, \\ \delta(\alpha) &= (\alpha^2 - K^2/N^2)^{\frac{1}{2}} \end{aligned}\right\} \tag{24}$$

and we choose those branches of the square roots which have positive real parts on the Fourier inversion contour. Substituting from (21), (22) and (23) into the Fourier transforms of the boundary conditions (14) and (15), we obtain the transmission conditions

$$\mathbf{D}_{\mathcal{W}}(\alpha)\mathbf{W} - \mathbf{D}_{\mathcal{F}}(\alpha)\mathbf{F} = \mathbf{D}_{\mathcal{F}}(-\alpha)\mathbf{F}_0, \tag{25}$$

$$\nu\mathbf{T}_{\mathcal{W}}(\alpha)\mathbf{W} - \mathbf{T}_{\mathcal{F}}(\alpha)\mathbf{F} = -\mathbf{T}_{\mathcal{F}}(-\alpha)\mathbf{F}_0, \tag{26}$$

respectively, where $\nu = \tilde{c}_{44}/\mu$,

$$\mathbf{F} = \begin{pmatrix} \Phi^P \\ \Psi^S \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} U^{qL} \\ U^{qT} \end{pmatrix}, \quad \mathbf{F}_0 = \begin{pmatrix} \frac{\phi_0}{2N\gamma(\alpha)}e^{N\gamma(\alpha)z'} \\ 0 \end{pmatrix}, \tag{27}$$

and the matrices $\mathbf{D}_{\mathcal{F}}, \mathbf{D}_{\mathcal{W}}, \mathbf{T}_{\mathcal{F}}$ and $\mathbf{T}_{\mathcal{W}}$ are given by

$$\mathbf{D}_{\mathcal{F}}(\alpha) = \begin{pmatrix} -i\alpha & -\delta(\alpha) \\ \gamma(\alpha) & -i\alpha \end{pmatrix}, \tag{28}$$

$$\mathbf{D}_{\mathcal{W}}(\alpha) = N^{-1}\begin{pmatrix} S_1^{qL}(\alpha,0) & S_1^{qT}(\alpha,0) \\ S_3^{qL}(\alpha,0) & S_3^{qT}(\alpha,0) \end{pmatrix}, \tag{29}$$

$$\mathbf{T}_{\mathcal{F}}(\alpha) = \begin{pmatrix} -2i\alpha\gamma(\alpha) & -\alpha^2 - \delta^2(\alpha) \\ \alpha^2 + \delta^2(\alpha) & -2i\alpha\delta(\alpha) \end{pmatrix}, \tag{30}$$

$$\mathbf{T}_{\mathcal{W}}(\alpha) = -i\alpha\mathbf{Q}(0)\mathbf{D}_{\mathcal{W}}(\alpha) + \mathbf{P}(0)\mathbf{D}'_{\mathcal{W}}(\alpha) \tag{31}$$

and

$$\mathbf{D}'_{\mathcal{W}}(\alpha) = N^{-2}\begin{pmatrix} \frac{dS_1^{qL}}{dz}(\alpha,0) & \frac{dS_1^{qT}}{dz}(\alpha,0) \\ \frac{dS_3^{qL}}{dz}(\alpha,0) & \frac{dS_3^{qT}}{dz}(\alpha,0) \end{pmatrix}. \tag{32}$$

It follows that there exist unique values for $\mathbf{W}$ and $\mathbf{F}$ provided

$$\det[\nu\mathbf{T}_{\mathcal{W}}(\alpha) - \mathbf{T}_{\mathcal{F}}(\alpha)\,[\mathbf{D}_{\mathcal{F}}(\alpha)]^{-1}\,\mathbf{D}_{\mathcal{W}}(\alpha)] \neq 0. \tag{33}$$

We expect that, for certain values of the elastic moduli, the left hand side of (33) will vanish for discrete real values of α. Such points correspond to Stonely type interfacial waves and our inversion contour must be indented so as to ensure that these radiate to infinity.

This completes the formal determination of the undetermined coefficients; Fourier inversion now yields

$$\phi(x,y;z') = \frac{-i\phi_0}{4}H_0^{(1)}(k\sqrt{(x-x')^2+z'^2}) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi^P(\alpha,z')e^{N\gamma(\alpha)z}e^{-iN\alpha x}\,dN\alpha, \quad (34)$$

$$\psi(x,y;z') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi^S(\alpha,z')e^{N\delta(\alpha)z}e^{-iN\alpha x}\,dN\alpha, \quad (35)$$

in $z<0$ and

$$\mathbf{u}(x,z;z') = \frac{1}{2\pi}\int_{-\infty}^{\infty}[U^{qL}(\alpha,z')\mathbf{S}^{qL}(\alpha,z) + U^{qT}(\alpha,z')\mathbf{S}^{qT}(\alpha,z)]e^{-iN\alpha x}\,dN\alpha,\ z>0, \quad (36)$$

where, as indicated earlier, the contour of integration is chosen so that the wave motion is outgoing at infinity. We note that it is possible to explicitly display all the various mode conversions by writing

$$\Phi^P(\alpha,z') = \phi_0\Phi^{PP}(\alpha)e^{N\gamma(\alpha)z'}, \quad (37)$$

$$\Psi^S(\alpha,z') = \phi_0\Psi^{PS}(\alpha)e^{N\gamma(\alpha)z'}, \quad (38)$$

$$U^{qL}(\alpha,z') = \phi_0U^{PqL}(\alpha)e^{N\gamma(\alpha)z'}, \quad (39)$$

and

$$U^{qT}(\alpha,z') = \phi_0U^{PqT}(\alpha)e^{N\gamma(\alpha)z'}, \quad (40)$$

where $U^{PqL}(\alpha)$ is the spectral amplitude of the quasi-longitudinal mode converted from the incident P-wave, etc.

4. DETERMINATION OF S^{qL} AND S^{qT}

Anticipating the possibility of "turning points" in the refracted field (total internal reflection), it is natural to seek a solution of (19) using the ansatz

$$\mathbf{U}(\alpha,z) = e^{iN\xi(\alpha,z)}\{F(N,\zeta(\alpha,z))\sum_{n=0}^{\infty}\frac{\mathbf{a}_n(\alpha,z)}{N^n} + G(N,\zeta(\alpha,z))\sum_{n=0}^{\infty}\frac{\mathbf{b}_n(\alpha,z)}{N^n}\}, \quad (41)$$

where

$$F(N,\zeta) = \mathrm{Ai}(N^{\frac{2}{3}}\zeta), \quad G(N,\zeta) = N^{\frac{1}{3}}\mathrm{Ai}'(N^{\frac{2}{3}}\zeta), \quad (42)$$

and the undetermined functions $\xi(\alpha,z)$, $\zeta(\alpha,z)$, $\mathbf{a}_n(\alpha,z)$ and $\mathbf{b}_n(\alpha,z)$ are all supposed slowly varying. Substituting (41) into (19) and equating coefficients of N^jF and N^jG to zero for $j=2,1,0,-1,-2,\ldots$ and using Airy's differential equation

$$\mathrm{Ai}''(Z) = Z\mathrm{Ai}(Z) \quad (43)$$

yields a recursive system of differential-algebraic relations for the unknown functions. To leading order we obtain

$$\mathbf{P}[-\xi'^2\mathbf{a}_0 + 2i\zeta\zeta'\xi'\mathbf{b}_0 + \zeta\zeta'^2\mathbf{a}_0] - (\mathbf{Q}+\mathbf{Q}^{\mathrm{T}})[-\alpha\xi'\mathbf{a}_0 + i\alpha\zeta\zeta'\mathbf{b}_0] + (\mathbf{I}-\alpha^2\mathbf{R})\mathbf{a}_0 = 0 \quad (44)$$

and

$$\mathbf{P}[-\xi'^2\mathbf{b}_0 + 2i\zeta'\xi'\mathbf{a}_0 + \zeta\zeta'^2\mathbf{b}_0] - (\mathbf{Q}+\mathbf{Q}^{\mathbf{T}})[-\alpha\xi'\mathbf{b}_0 + i\alpha\zeta'\mathbf{a}_0] + (\mathbf{I}-\alpha^2\mathbf{R})\mathbf{b}_0 = 0, \tag{45}$$

where the prime denotes $\frac{d}{dz}$. Multiplying (44) by $\zeta^{-\frac{1}{2}}$ and (45) by $\zeta^{\frac{1}{2}}$ and adding and subtracting the resulting identities we find that

$$[\mathbf{P}(\Lambda^{\pm})^2 - \alpha(\mathbf{Q}+\mathbf{Q}^{\mathbf{T}})\Lambda^{\pm} - (\mathbf{I}-\alpha^2\mathbf{R})]\mathbf{A}_0^{\pm} = 0, \tag{46}$$

where

$$\Lambda^{\pm} = \xi'(\alpha,z) \mp i\zeta^{\frac{1}{2}}(\alpha,z)\zeta'(\alpha,z) \tag{47}$$

and

$$\mathbf{A}_0^{\pm} = \zeta^{-\frac{1}{4}}\mathbf{a}_0(\alpha,z) \pm \zeta^{\frac{1}{4}}(\alpha,z)\mathbf{b}_0(\alpha,z). \tag{48}$$

A necessary and sufficient conditon for (46) to have a non-trivial solution for $\mathbf{A}_0^{\pm}$ is

$$\det[\mathbf{P}(\Lambda^{\pm})^2 - \alpha(\mathbf{Q}+\mathbf{Q}^{\mathbf{T}})\Lambda^{\pm} - (\mathbf{I}-\alpha^2\mathbf{R})] = 0, \tag{49}$$

a quartic equation for the determination of $\Lambda^{\pm}$. Proceeding to the next order in the expansion we find similarly that

$$[\mathbf{P}(\Lambda^{\pm})^2 - \alpha(\mathbf{Q}+\mathbf{Q}^{\mathbf{T}})\Lambda^{\pm} - (\mathbf{I}-\alpha^2\mathbf{R})]\mathbf{A}_1^{\pm} = \mathbf{f}_1^{\pm}, \tag{50}$$

where

$$\begin{aligned} \mathbf{f}_1^{\pm}(\alpha,z) &= -i\mathbf{P}(\mathbf{A}_0^{\pm})'\Lambda^{\pm} - i(\mathbf{P}\mathbf{A}_0^{\pm}\Lambda^{\pm})' + i\alpha\mathbf{Q}'\mathbf{A}_0^{\pm} + i\alpha(\mathbf{Q}+\mathbf{Q}^T)\mathbf{A}_0^{\pm} \\ &+ i\frac{\zeta'}{4\zeta}[\mathbf{P}\mathbf{A}_0^{\mp}(\Lambda^+ + \Lambda^-) - \alpha(\mathbf{Q}+\mathbf{Q}^T)\mathbf{A}_0^{\mp}] \end{aligned} \tag{51}$$

and so a necessary condition for there to be a solution of (50) is that $\mathbf{f}_1^{\pm}$ is orthogonal to $\mathbf{A}_0^{\pm}$, ie.

$$(\mathbf{f}_1^{\pm})^T\mathbf{A}_0^{\pm} = 0. \tag{52}$$

Adding (52) to the transpose of itself, using (46) and a symmetry argument yields the simple uncoupled conservation equations

$$\left[(\mathbf{A}_0^{\pm})^T\left(2\mathbf{P}\Lambda^{\pm} - \alpha(\mathbf{Q}+\mathbf{Q}^T)\right)\mathbf{A}_0^{\pm}\right]' = 0. \tag{53}$$

Now let $\mathbf{d}_0^{\pm}$ be unit vectors in the direction of the leading order amplitudes $\mathbf{A}_0^{\pm}$, then the general solutions of (53) are

$$\mathbf{A}_0^{\pm} = \mathbf{d}_0^{\pm}(\alpha,z)\left[\frac{\Lambda^{\pm}}{1+(\Lambda^{\pm})^2P_0^{\pm}(\theta(z)) - \alpha^2R_0^{\pm}(\theta(z))}\right]^{\frac{1}{2}}, \tag{54}$$

where

$$P_0^{\pm}(\theta(z)) = (\mathbf{d}_0^{\pm})^T\mathbf{P}\mathbf{d}_0^{\pm}, \quad R_0^{\pm}(\theta(z)) = (\mathbf{d}_0^{\pm})^T\mathbf{R}\mathbf{d}_0^{\pm}. \tag{55}$$

We assume that, in principle at least, the higher order amplitude coefficients in the expansion (41) may be obtained by repeated application of the orthogonality condition.

Now for real values of α, the characteristic equation (49) can only have real roots or roots occuring in complex conjugate pairs. It follows that, without loss of generality, we may choose $\xi(\alpha,z)$ and $\zeta(\alpha,z)$ to be real valued functions for all real α. If $\zeta(\alpha,z)$ is of one sign and bounded away from zero for all α, z then we may replace the Airy functions by their asymptotic expansions for large argument. We find that

$$\mathbf{U}(\alpha,z) = \frac{1}{2\sqrt{\pi}N^{\frac{1}{6}}}\mathbf{A}_0^- \exp iN\chi^-(\alpha,z)(1+O(N^{-1})), \quad \zeta(\alpha,z) > 0, \tag{56}$$

$$\mathbf{U}(\alpha,z) = \frac{1}{2\sqrt{\pi}N^{\frac{1}{6}}}\left[\mathbf{A}_0^- \exp iN\chi^-(\alpha,z) - i\mathbf{A}_0^+ \exp iN\chi^+(\alpha,z)\right](1+O(N^{-1})), \quad \zeta(\alpha,z) < 0, \tag{57}$$

where

$$\chi^{\pm} = \left. \begin{matrix} \xi \mp i\frac{2}{3}\zeta^{\frac{3}{2}} & \zeta > 0 \\ \xi \mp \frac{2}{3}(-\zeta)^{\frac{3}{2}} & \zeta < 0 \end{matrix} \right\}. \tag{58}$$

On the other hand, if ζ changes sign as z varies for a given α, i.e. the differential equation (19) has a turning value, then it is clear that this approximation is not uniformly valid near $\zeta = 0$. Indeed, it is readily seen from (48) that the denominator in (54) also vanishes as $\zeta \to 0$. However, if we choose

$$\chi^{\pm}(\alpha, z) = \int_{z^*}^{z} \Lambda^{\pm}(\alpha, t)\, dt, \tag{59}$$

where z^* is chosen for each value of α so that

$$\zeta(\alpha, z^*) = 0, \tag{60}$$

then it is clear that the coefficients $\mathbf{a}_0, \mathbf{b}_0$ remain bounded and the approximation (41) is uniformly valid through the turning values. We conclude that it is in fact *essential* to choose ξ, ζ to be real for real values of α for the latter property to hold. Further, it is also necessary to choose $\chi^-(\alpha, z)$ to have positive imaginary part on the Fourier inversion contour for the integrals to converge. It turns out that the latter condition is also sufficient to ensure that the waves corresponding to a particular propagating mode radiate outwards at infinity. To complete the formal solution of the refraction problem we need to choose two linearly independent solutions of (19) which we have nominally labelled $\mathbf{S}^{qL}$ and $\mathbf{S}^{qT}$. We shall describe this process for the refracted quasi-longitudinal wave in the following section.

5. THE REFRACTED QUASI-LONGITUDINAL WAVES

The complete detailed evaluation of the refracted and reflected field for an incident compression and shear source is given in Abrahams & Wickham, (1991b). Here we outline the construction of the refracted "quasi-compression waves" $\mathbf{u}^{PqL}$ for the current problem; we have from (36)

$$\mathbf{u}^{PqL}(x, z; z') = \frac{N\phi_0}{2\pi} \int_{-\infty}^{\infty} U^{PqL}(\alpha) \mathbf{S}^{qL}(\alpha, z) e^{N\gamma(\alpha)z' - i\alpha N x}\, d\alpha, \quad z > 0. \tag{61}$$

The structure of this integral is in fact exactly analogous to the corresponding result studied by the authors (Abrahams & Wickham, 1991a) for the case of horizontally polarised shear waves. Following the procedure used there, it may be shown that, for a given value of (x, z), the contour of integration may be deformed away from the turning values of the differential equation (19) so that the arguments of the Airy functions are uniformly large over the whole of the integration range. This justifies replacing (41) by the approximations (56) and (57) as appropriate and then restoring the contour of integration to the whole of the real α-axis. Over that part of the integration range where $\zeta(\alpha, z)$ is negative it turns out that the determinant in (33) and hence the amplitude factor U^{PqL} is not slowly varying as $N \to \infty$! However, as in Abrahams & Wickham (1991a), it is possible to expand the inverse of this determinant in a power series of rapidly varying exponentials so that the resulting expression for $\mathbf{u}^{PqL}$ is an infinite sum of classical diffraction integrals. The first of these is of the form

$$\mathbf{u}_0^{PqL}(x, z; z') = \frac{\phi_0}{2\pi} \int_{-\infty}^{\infty} U_0^{PqL}(\alpha) \mathbf{A}_0^-(\alpha, z) e^{iN\phi_{qL}^-(\alpha, x, z; z')}\, d\alpha, \tag{62}$$

where the amplitude factor U_0^{PqL} varies slowly with α and the phase is given by

$$\phi_{qL}^-(\alpha, x, z; z') = \int_0^z \Lambda_{qL}^-(\alpha, Z)\, dZ - i\gamma(\alpha) z' - \alpha x. \tag{63}$$

The integral in (62), and all the subsequent terms in the series, are now in a form suitable for applying Kelvin's method of stationary phase. However, all this assumes that we have already decided which eigensolution of (46) corresponds to Λ_{qL}^{-}.

If we focus attention on a particular station (x, z) in W, then the motion corresponding to (62) is the superposition of Fourier contributions over a continuous spectrum of the x-component of the slowness vector. To see this, consider a neighbouring point $(x+\Delta x, z+\Delta z) \equiv \mathbf{r}+\Delta\mathbf{r}$. There, we have a contribution to the Fourier integral which, to leading order in $\Delta\mathbf{r}$, is given by

$$U_0^{PqL}(\alpha)\mathbf{A}_0^{-}(\alpha, z)e^{iN\phi_{qL}^{-}}e^{iN\mathrm{grad}\phi_{qL}^{-}.\Delta\mathbf{r}-i\omega t}(1+O(\Delta x, \Delta z)), \tag{64}$$

where, for completeness we have included the time dependence in this expression. By re-writing the increment in the phase as

$$N\mathrm{grad}\phi_{qL}^{-}.\Delta\mathbf{r}-\omega t = KqL(\mathbf{n}_{qL}.\Delta\mathbf{r}-c_{qL}t) = \omega(\mathbf{s}.\Delta\mathbf{r}-t), \tag{65}$$

we are able to identify the local wavenumber, K_{qL} phase normal $\mathbf{n}_{qL}$, phase speed c_{qL} and slowness vectors according to

$$K_{qL} = N\sqrt{\alpha^2+\left(\Lambda_{qL}^{-}\right)^2}, \tag{66}$$

$$\mathbf{n}_{qL} = (-\alpha, \Lambda_{qL}^{-}(\alpha, z))/\sqrt{\alpha^2+\left(\Lambda_{qL}^{-}\right)^2}, \tag{67}$$

$$c_{qL} = \omega/K_{qL} \tag{68}$$

and

$$\mathbf{s} = (-\alpha, \Lambda_{qL}^{-}(\alpha, z))\sqrt{\frac{\rho}{\bar{c}_{44}}}, \tag{69}$$

respectively. The wave kinematics only depend on the local material properties, which, are constant relative to the grain coordinates. Thus

$$\bar{s}_i = a_{ij}s_j \tag{70}$$

But $(\alpha, \Lambda_{qL}^{-}(\alpha, z))$ satisfies the characteristic equation (49) and so, using (70) to eliminate α and $\Lambda_{qL}^{-}(\alpha, z)$, we obtain an equation independent of θ, namely

$$\bar{s}_z^4\bar{c}_{33}+\bar{s}_z^2(\bar{s}_x^2C_0-\bar{c}_{33}-1)+\bar{c}_{11}\bar{s}_x^4-\bar{s}_x^2(\bar{c}_{11}+1)+1=0, \tag{71}$$

with

$$C_0 = \bar{c}_{33}\bar{c}_{11}-\bar{c}_{13}^2-2\bar{c}_{13}. \tag{72}$$

Equation (71) is precisely the slowness surface for a homogeneous transversely isotropic material in plane strain with the zonal axis in the 3-direction, Musgrave (1970), pp 96–101. This quartic surface separates trivially into two sheets, which, when expressed in polar coordinates

$$\bar{s}_x = \bar{s}\sin\varphi, \qquad \bar{s}_z = \bar{s}\cos\varphi, \tag{73}$$

are given by

$$\frac{\bar{s}^2}{\bar{c}_{44}} = \frac{\mathcal{A}\pm\sqrt{\mathcal{A}^2-4\mathcal{B}(\varphi)}}{2\mathcal{B}(\varphi)}, \tag{74}$$

where $\mathcal{A}$ is the trace of the plane-strain Hookean tensor, ie.

$$\mathcal{A} = \bar{c}_{33}+\bar{c}_{11}+2\bar{c}_{44} \tag{75}$$

and

$$B(\varphi) = \bar{c}_{44}\bar{c}_{33}\cos^4\varphi + C\sin^2\varphi\cos^2\varphi + \bar{c}_{11}\bar{c}_{44}\sin^4\varphi, \tag{76}$$

with

$$C = \bar{c}_{33}\bar{c}_{11} + \bar{c}_{13}^2 - 2\bar{c}_{13}\bar{c}_{44}. \tag{77}$$

The sheet corresponding to the greater phase speed is associated with the quasi-longitudinal motion, ie.

$$(c_{qL})^2 = \frac{\bar{c}_{44}}{\rho}\frac{1}{\bar{s}^2} = \frac{\bar{c}_{44}}{\rho}\frac{2B(\varphi)}{A - \sqrt{A^2 - 4B(\varphi)}} \tag{78}$$

Thus, given a value of z and α_0 such that equation (49) has a real solution $\Lambda_{qL}^-(\alpha_0, z)$, we may calculate the corresponding $\bar{s}$ and φ using equations (70) and (73) and hence identify a point on one of the slowness sheets (74). The direction of energy flux at that point is normal to the slowness surface, Musgrave (1970), and therefore we are also able to use (74) to discriminate which of the real solutions of the eigenvalue problem correspond to outgoing waves at infinity.

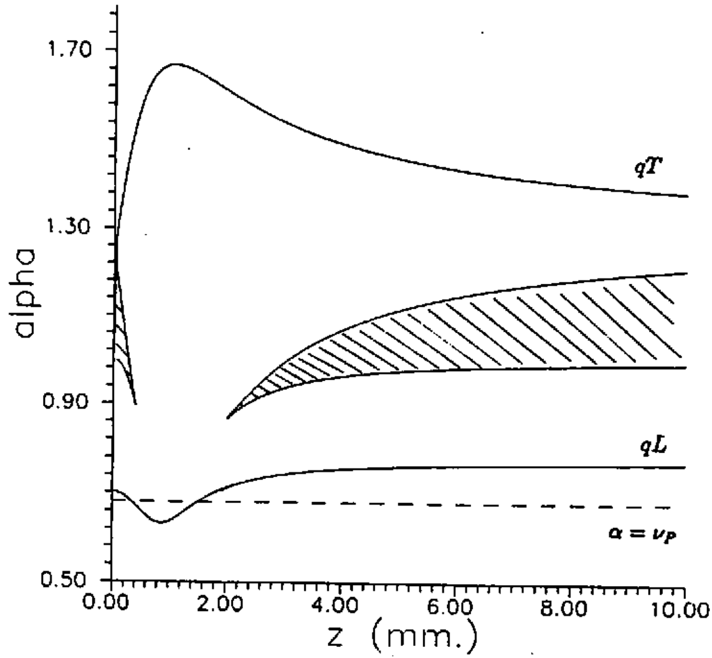
Having settled the question of which root of (49) provides the required wave field we are then able to proceed to the determination of the leading order contributions to (61). The condition for a point of stationary phase is

$$\int_0^z \frac{\partial \Lambda_{qL}^-}{\partial \alpha}(\alpha, Z)\, dZ + \frac{\alpha}{\sqrt{\nu_P^2 - \alpha^2}} z' - x = 0, \tag{79}$$

where $\nu_P = k/N$. Evidently, this may only have a real solution for α if $\Lambda_{qL}^-(\alpha, z)$ is real over the whole of the integration range in (79) and $|\alpha| < \nu_P$. We note that the existence of a real stationary phase point at a given station is precisely equivalent to there being a refracted ray (in the sense of geometrical acoustics) passing through that station. The characteristic value Λ_{qL}^- can only become complex if, for a given value of α, Z passes through a turning point of the differential equation (19). The turning points correspond to *the zeros of the discriminant of the quartic equation* (49) and it is proved in Abrahams & Wickham (1991b) that a necessary and sufficient condition for some refracted rays to be totally internally reflected is that *ν_P is greater than the least zero of that discriminant*. We also show that total internal reflection implies the formation of an infinite set of caustic surfaces. The turning rays undergo an infinite sequence of reflections and refractions at the interface and each bundle of reflected rays envelope a caustic. These contributions to the wave field may be quantified by the "higher order" terms in the infinite series of integrals representing (61).

The analysis outlined here has highlighted the importance of the discriminant of (49) in determining the qualitative structure of the refracted field. It is particularly helpful in an inhomogeneous medium to chart the locus of zeros of the discriminant in the (α, z)-plane. We call such a graph the phase indicator; figure 2 provides an example where the grain structure is a parabolic profile and the material constants pertain to fine-grained austenitic steel for $\mathcal{F}$ and coarse-grained austenite for the weld material $\mathcal{W}$. Evidently, the locus of zeros define an open set in the (α, z)-plane given by a finite number of piecewise continuous boundaries. These curves delineate various regions in which the roots of (49) vary as analytic functions of α and z, and within which the number of real roots is unchanging. For points (α, z) below the lowest of these "modal boundaries" as we call them, (49) has four real roots corresponding to forward (z-increasing) and backward propagating quasi transverse and longitudinal waves. Above the top modal boundary (labelled qT), all the roots are complex, while in the region between the qL and qT curves there are generally only two real roots corresponding to quasi-transverse waves. The exceptions are in the shaded regions where the material admits the possibility of a second pair of propagating transverse waves. This feature is discussed at length in Abrahams & Wickham (1991b); here we focus attention on the qL modal boundary. Firstly it is clear that there are no refracted quasi-longitudinal rays for incident rays striking the interface with an x-component of its slowness vector greater than $\nu_P\sqrt{\bar{c}_{44}/\rho}$. This of course is the classical critical angle of refraction for the

Figure 2: The phase indicator graph for the parabolic profile $\theta(z) = \arctan z$. The material constants for this example are $\bar{c}_{11} = 26.28Nm^{-2}$, $\bar{c}_{12} = 9.73Nm^{-2}$, $\bar{c}_{13} = 14.5Nm^{-2}$, $\bar{c}_{33} = 21.6Nm^{-2}$, and $\bar{c}_{44} = 12.9Nm^{-2}$ and correspond to coarse grained austenitic steel. The contrast parameter for compression waves is taken to be $\nu_P = 0.67$; the dotted line is $\alpha = \nu_P$.



interface problem. However, we note that a striking feature of the qL-modal curve is that, at a certain depth into $\mathcal{W}$, it dips below the critical line $\alpha = \nu_P$. In Abrahams & Wickham (1991b) it is shown that whenever this occurs then there will always be a bundle of refracted rays totally internally reflected. It follows from the above example that this phenomenon appears to be relevant to the ultrasonic inspection of austenitic welds using compression wave probes.

6. CONCLUDING REMARKS

In this article we have outlined an exact analytical approach to the quantitative and qualitative understanding of the refraction of ultrasound from isotropic sources at the interface with a homogeneous isotropic solid and an inhomogeneously oriented transversely anisotropic material. The problem considered is canonical in that it focusses attention on just one-dimensional variations in the orientation of the zonal axis. This leads to a tractable, albeit complicated, mathematical scattering problem which we solve rigorously in the high frequency limit. The solution has pin-pointed a useful graphical tool in the phase indicator graph, which, in conjunction with the slowness surface for the anisotropic material, may be used to effectively predict the qualitative structure of the refracted field. Our solutions may be used as benchmarks to compare with heuristic numerical techniques such as that described by Ogilvy (1986) and formal ray analysis. The latter provides the best hope for quantifying refraction through more general oriented materials and some progress to this end will be reported in Norris & Wickham (1991). The latter uses the Gaussian beam summation method which is fully discussed in White, Norris, Bayliss & Burridge (1987).

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