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ANALYSIS OF PIEZOELECTRIC TRANSDUCERS COMBINING FINITE ELEMENTS AND INTEGRAL REPRESENTATIONS

H. Jensen and S. Krenk

Risø National Laboratory, Roskilde, Denmark

INTRODUCTION

Piezoelectric materials are widely used in ultrasonic transducers, but owing to the complexity of the full constitutuve equations for these materials simplifying assumptions are often introduced in design calculations. However, the finite element method is sufficiently general to deal with the full equations of piezoelectricity as well as complicated geometries [1,2]. The accuracy of such calculations depends strongly on the number of elements used to model the problem, and in vibration problems the element size must be small compared with any characteristic wave length as well as any geometric length scale. This leads to a large number of elements even for simple geometries. As piezoelectric devices are often operated near resonance, the electric field should also be modelled explicitly via an electrical degree of freedom at each node. At, or near, resonance damping is also important.

In the following the finite element method is used to analyse axisymmetric vibrations of piezoelectric ceramics, where it is used that the piezoelectric ceramics and crystals of class 6mm are transversely isotropic. The time factor $\exp(-i\omega t)$ is omitted from the formulas. The transducer, here in the form of a simple disc, is modelled by triangular ring elements and joined with a fluid half-space modelled by an integral representation of the acoustic field. The matching procedure makes use of a least square fit. In addition to the radiation damping material damping can be included via complex material constants. The integral representation in the half-space is selected with a special view to explicit forms of the axial field and the far field.

THE PIEZOELECTRIC RING ELEMENT

Introduce a cylindrical coordinate system with coordinates $x_1 = r$ and $x_3 = z$. When Greek indices take the values 1 and 3 an axisymmetric state of deformation is given by the displacement vector $u_\beta(x_\alpha)$, and $\phi(x_\alpha)$ is an axisymmetric electric potential. In compressed vector notation the relevant components of the strain vector are

$$S_j = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix} = \begin{Bmatrix} \partial u_1/\partial x_1 \\ u_1/x_1 \\ \partial u_3/\partial x_3 \\ \partial u_1/\partial x_3 + \partial u_3/\partial x_1 \end{Bmatrix} \tag{1}$$

The electric field vector is

$$E_\beta = \begin{Bmatrix} E_1 \\ E_3 \end{Bmatrix} = - \begin{Bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_3 \end{Bmatrix} \tag{2}$$

With the choice of S_i and E_β as the independent variables the constitutive equations are [3]

$$T_i = c_{ij}S_j - e_{i\alpha}E_\alpha \tag{3}$$

$$D_\alpha = e_{j\alpha}S_j + \varepsilon_{\alpha\beta}E_\beta \tag{4}$$

T_i is the stress vector, D_α is the electric charge vector and c_{ij}, $e_{i\alpha}$ and $\varepsilon_{\alpha\beta}$ are material parameter matrices. Summation over repeated indices is implied.

In the absence of external and inertial forces the Lagrangian is [1]

$$L = \frac{1}{2}\int_V \{S_i c_{ij} S_j - 2S_i e_{i\alpha} E_\alpha - E_\alpha \varepsilon_{\alpha\beta} E_\beta\}\, dV \tag{5}$$

From this integral the stiffness matrix of the finite element is derived along the lines of [4]. The elements are circular rings with triangular cross sections, and the shape functions are linear with nodes in the corners.

Each element contributes with a stiffness and a mass matrix that relate nodal displacements and electric potential on one side to nodal forces F_α^j and electric charge Q^j on the other side.

$$\begin{Bmatrix} F_1^j \\ F_3^j \\ Q^j \end{Bmatrix} = \left(\begin{bmatrix} K_{11}^{jk} & K_{13}^{jk} & K_{14}^{jk} \\ K_{31}^{jk} & K_{33}^{jk} & K_{34}^{jk} \\ K_{41}^{jk} & K_{43}^{jk} & K_{44}^{jk} \end{bmatrix} - \omega^2 \begin{bmatrix} M^{jk} & 0 & 0 \\ 0 & M^{jk} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} u_1^k \\ u_3^k \\ \phi^k \end{Bmatrix} \tag{6}$$

The submatrices can be expressed in explicit form in terms of the volume V, the cross sectional area A, the mass density ρ and the coordinates x_α^j of the corners. In the stiffness matrix it is convenient to replace the corner coordinates by the coordinates of the sides considered as vectors, $\ell_\alpha^1 = x_\alpha^3 - x_\alpha^2$ etc. The calculation makes use of triangular area coordinates defined by $\lambda_j = A_j/A$, where A_j is the area of the triangle obtained by substituting the point in question for the corner j. With a single exception all terms are of algebraic form. The exceptional term is

$$J^{jk} = 2\pi \int_A \frac{\lambda_j \lambda_k}{x_1}\, dA \tag{7}$$

This integral has been evaluated in terms of the logarithmic function for arbitrary triangles in [4].

The explicit form of the submatrices in (6) is

$$K_{11}^{jk} = \frac{V}{4A^2}\{c_{11}\ell_3^j\ell_3^k + c_{55}\ell_1^j\ell_1^k\} - \frac{\pi}{3}c_{12}\{\ell_3^j + \ell_3^k\} + c_{11}J^{jk} \tag{8}$$

$$K_{13}^{jk} = \frac{-V}{4A^2}\{c_{13}\ell_3^j\ell_1^k + c_{55}\ell_1^j\ell_3^k\} + \frac{\pi}{3}c_{13}\ell_1^k \tag{9}$$

$$K_{14}^{jk} = \frac{-V}{4A^2}\{e_{31}\ell_3^j\ell_1^k + e_{15}\ell_1^j\ell_3^k\} + \frac{\pi}{3}e_{31}\ell_1^k \tag{10}$$

$$K_{33}^{jk} = \frac{V}{4A^2}\{c_{55}\ell_3^j\ell_3^k + c_{33}\ell_1^j\ell_1^k\} \tag{11}$$

$$K_{34}^{jk} = \frac{V}{4A^2}\{e_{15}\ell_3^j\ell_3^k + e_{33}\ell_1^j\ell_1^k\} \tag{12}$$

$$K_{44}^{jk} = \frac{-V}{4A^2}\{\varepsilon_{11}\ell_3^j\ell_3^k + \varepsilon_{33}\ell_1^j\ell_1^k\} \tag{13}$$

$$M^{jk} = (1 - \delta_{jk})\frac{\pi A\rho}{30}\{x_1^j + x_1^k + \sum_{m=1}^{3} x_1^m\} \tag{14}$$

δ_{jk} is Kronecker's delta.

THE HALF-SPACE

Within linear acoustics the velocity field and the pressure can be derived from a scalar potential ψ.

$$p = p(x_\alpha) = \rho\dot{\psi} = -i\omega\rho\psi(x_\alpha) \tag{15}$$

where ρ is the mass density.

Let an axisymmetric normal velocity v be prescribed on the surface of the half-space $x_3 < 0$. When $v = 0$ outside a circular area with radius a, the pressure can be represented as [5]

$$p(x_\alpha) = -\frac{i\omega\rho}{2\pi}\int_0^{2\pi}\int_0^a v(\xi)\frac{e^{ikr}}{r}\xi d\xi\, d\theta \tag{16}$$

where $r = (x_3^2 + x_1^2 + \xi^2 - 2x_1\xi\cos\theta)^{\frac{1}{2}}$, $k = \omega/c$ is the wavenumber and c is the speed of sound.

The stiffness properties of the half-space are determined from the relation between $v(\xi)$ and the pressure $p(x_1)$ for $x_3 = 0$. It is convenient to expand $v(x_1)$ and $p(x_1)$ in terms of orthogonal polynomials.

$$\begin{Bmatrix} p(x_1) \\ v(x_1) \end{Bmatrix} = \sum_{n=0}^{N} \begin{Bmatrix} p_n \\ v_n \end{Bmatrix} P_n(1 - 2x_1^2/a^2) \tag{17}$$

$P_n(\ )$ is the Legendre polynomial of degree n. The expansion coefficients follow from the orthogonality relation of the Legendre polynomials.

$$\begin{Bmatrix} p_n \\ v_n \end{Bmatrix} = (2n+1)\frac{2}{a^2}\int_0^a \begin{Bmatrix} p(\xi) \\ v(\xi) \end{Bmatrix} P_n(1-2\xi^2/a^2)\,\xi d\xi \tag{18}$$

The relation between the expansion coefficients is found by substitution of the expansion (17) followed by use of the orthogonality relation. The result is in the form

$$p_m = -2i\ \rho c\ (2m+1)\ Z_{mn} v_n \tag{19}$$

where summation over n is implied, and the matrix Z_{mn} is

$$Z_{mn} = \frac{k^2}{2\pi a^2}\int_0^{2\pi}\int_0^a\int_0^a P_m(1-2\xi^2/a^2)P_n(1-2\eta^2/a^2)\frac{e^{ikr}}{kr}\,\xi d\xi\,\eta d\eta\,d\theta \tag{20}$$

where $r = (\xi^2+\eta^2-2\xi\eta\cos\theta)^{\frac{1}{2}}$. The exponential factor is the point source solution in three dimensions. It can be represented as a Hankel transform (see e.g. 6.554 in [6]). Upon substitution of this representation in (20) the integration can be carried out by use of Neumann's addition formula.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ikr}}{kr}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{J_o(kru)}{(u^2-1)^{\frac{1}{2}}}u\,du\,d\theta = \int_0^{\infty}\frac{J_o(k\xi u)J_o(k\eta u)}{(u^2-1)^{\frac{1}{2}}}u\,du \tag{21}$$

The Bessel functions and the Legendre polynomials combine into known Hankel transforms (see e.g. 7.251 in [6]) whereby (20) takes the form of a single integral.

$$Z_{mn} = \int_0^{\infty}\frac{J_{2m+1}(kau)J_{2n+1}(kau)}{u(u^2-1)^{\frac{1}{2}}}du \tag{22}$$

Direct numerical evaluation of this integral is difficult due to the oscillatory behaviour of the integrand. However, the integration path can be changed to the interval [0;1] plus the imaginary axis, and the latter is easily transformed into a finite interval.

THE MATCHING PROCEDURE

The normal velocity of the face of the transducer is given by

$$v(x_1) = -i\omega\sum_m u_3^m N_m(x_1) \tag{23}$$

where u_3^m is the displacement component of node number m in the x_3 direction, and $N_m(x_1)$ is the shape function associated with u_3^m.

The transducer is matched to the half-space by a least square fit. Use of (18) gives

$$v_n = - i\omega L_{nm} u_3^m \tag{24}$$

where summation over the nodes m is implied and the matrix L_{nm} is

$$L_{nm} = (2n+1)\frac{2}{a^2}\int_0^a N_m(\xi)P_n(1-2\xi^2/a^2)\,\xi d\xi \tag{25}$$

The pressure is integrated to give the nodal forces

$$F_3^l = 2\pi \int_0^a N_l(\xi)p(\xi)\,\xi d\xi = \frac{\pi a^2}{2n+1} L_{ln}^T p_n \tag{26}$$

The influence of the half-space on the transducer is then given as a stiffness matrix found by combining (19), (24) and (26).

$$C_{kl} = - 2\pi \rho c \omega a^2 L_{kn}^T z_{nm} L_{ml} \tag{27}$$

When the local matrices have been combined and the resulting system of equations has been solved for the nodal displacements and electric potentials, the expansion coefficients v_n follow from (24). The particular choice of expansion (17) leads to a simple, explicit form of the pressure on the axis as demonstrated in [5]. For $x_1 = 0$

$$p(x_3) = \frac{1}{2}\rho c\,(ka)^2 \sum_{n=0}^{N} v_n\, j_n(kz)\, h_n^{(1)}(ky) \tag{28}$$

$j_n(\ )$ and $h_n^{(1)}(\ )$ are the spherical Bessel and Hankel Functions, respectively. The variables z and y are defined by

$$2z = (x_3^2 + a^2)^{\frac{1}{2}} - x_3 \quad , \qquad 2y = (x_3^2 + a^2)^{\frac{1}{2}} + x_3 \tag{29}$$

Also the far field takes a simple form in terms of the expansion coefficients v_n.

$$p(x_\alpha) \sim \frac{e^{ikr}}{r}\, a^2 \sum_{n=0}^{N} (-1)^n v_n \frac{J_{2n+1}(ka\sin\theta)}{ka\sin\theta} \tag{30}$$

where $r = (x_1^2 + x_3^2)^{\frac{1}{2}}$ and $\sin\theta = x_1/r$.

EXAMPLE

A computer program has been written according to the present theory, and it has been used to investigate the vibrations of a disc of PZT 65/35 [7] with a diameter to thickness ratio of 10. The thickness was 1 mm. Based on the material parameters from [7] the thickness frequency of an infinite plate is 2.06 MHz. The program determined the following eigenfrequencies for free vibrations (in MHz)

0.25	0.66	1.02	1.33	1.55	1.66	1.80	1.98
2.00	2.02	2.06	2.14	2.21			

The first seven correspond closely to products of a Bessel function for the radial variation and a trigonometric thickness variation. Deviations appear to be confined to the curved boundary. The following two modes are difficult to classify, while the the tenth mode is dominated by shear and the eleventh mode appears to be the thickness mode. Fig.1 shows contour plots of electric potential, axial and radial displacements together with the deformed geometry of a quarter of the disc in modes four and ten, i.e. a simple mode and the shear mode.

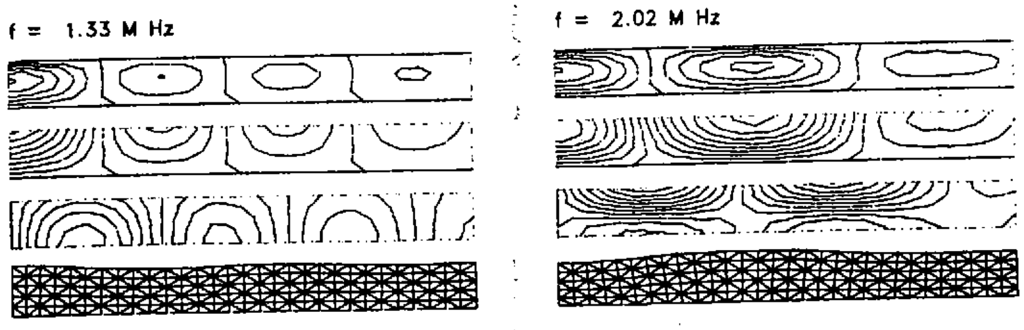

Fig.1. Contour plots of ϕ, u_3 and u_1 and deformed geometry of a quarter of the disc in free vibrational modes four and ten.

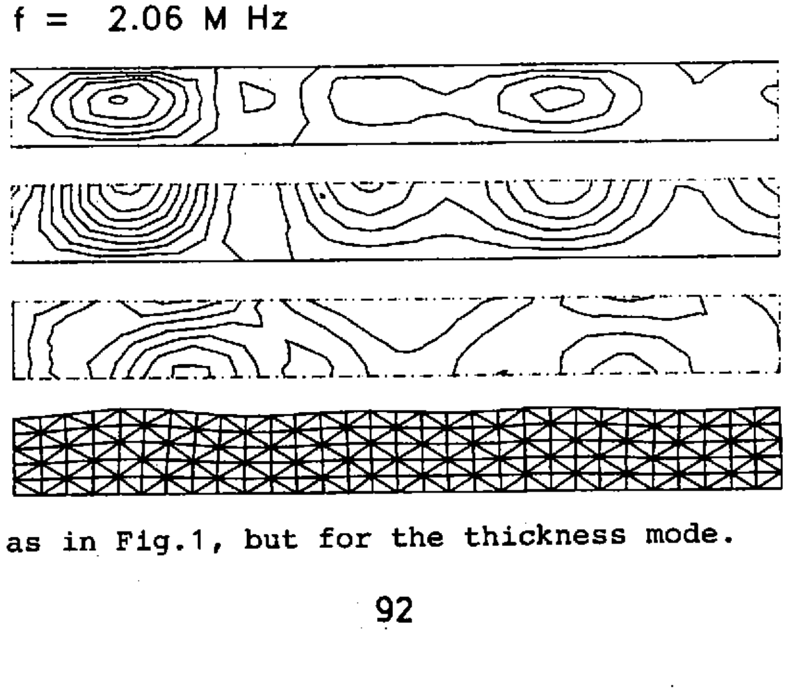

Fig.2. Data as in Fig.1, but for the thickness mode.

Fig.2 shows similar results for the thickness mode in some more detail. It is clearly seen that the faces of the transducer do not remain plane. However, the deviation from piston behaviour does not influence the central lobe appreciably. This is illustrated in Fig.3 showing the radiated pressure field on the axis and the directional pattern in the far field, when the transducer is radiating in water.

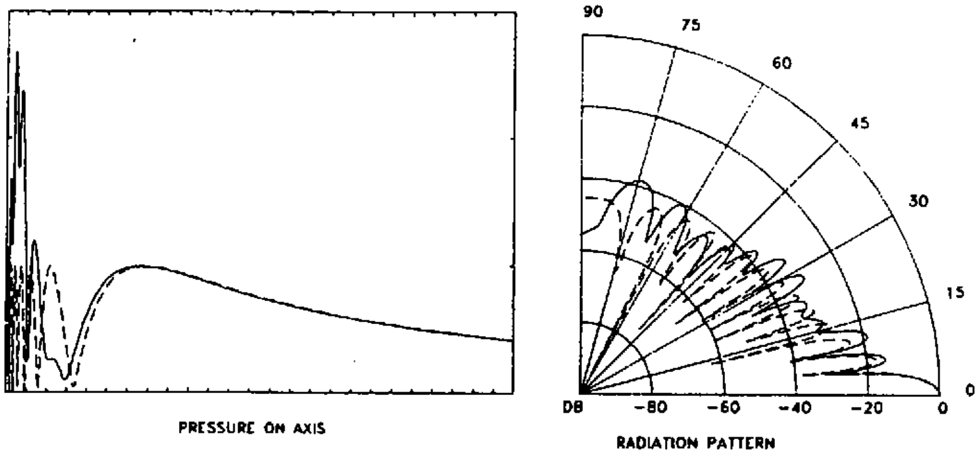

Fig.3. Axial pressure distribution and directional pattern for radiation in water at 2.06 MHz. Numerical solution in full line and piston solution in dashed line.

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