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THE RELIABILITY ANALYSIS OF STOCHASTIC NONLINEAR VIBRATION SYSTEM WITH FAILURE INTERACTION

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The reliability analysis of stochastic nonlinear vibration system considering failure interaction under incomplete probabilistic information is performed. Firstly, the response of the stochastic vibration system is calculated by stochastic perturbation technique. Then, the performance functions of the vibration system are defined according to the first passage theory. The reliability of marginal probability of failure is estimated based on the saddlepoint approximation and obtained moments of performance functions. The copula theory is adopted as a mathematical tool to model the dependency characteristic between each failure mode. The joint probability distribution functions between failure modes are established by using Gaussian copula functions. The undetermined coefficients in copula function are estimated using statistical approach. The method can be used for the system reliability analysis of stochastic nonlinear vibration system with dependent failure modes.
Keywords: reliability, stochastic vibration system, failure interaction, copula

1. Introduction

The reliability of engineering structures is the most important objective of structure design. However, uncertainties in material properties and structural geometry that due to the manufacturing error or structure complexities may influence the reliability of the structure. These uncertainties can be described as the mass, damping and stiffness. Generally, the uncertainties are considered as random variables or stochastic process. The study on the reliability of uncertain systems is a key issue during the design process. With the help of the reliability analysis, one can define the acceptable tolerances on structures and determine the range of the system parameters for safe operations.

During the last decades, the reliability analysis methods for random structural systems are have been greatly developed [1-5]. However, the research on the random dynamic structural systems, that are far more complicated than static state systems, is still in the initial stage. Therefore, when it comes to the reliability analysis of multi-degree-of freedom nonlinear vibration systems with dependent failure modes, few literatures can be found.

This paper presents a copula-based method for stochastic multi-degree-of-freedom nonlinear vibration systems with dependent failure modes. The random responses of the stochastic vibration systems are formulated. The first four moments of the limit state function are derived by using random perturbation technique. Unknown distribution function of the limit state function is determined by the saddlepoint approximation method. The joint probability between pairs of failure mode is calculated by using the Gaussian copula. The system reliability of multi-degree-of-freedom nonlinear random structure vibration systems is then estimated with the reliability bound method.

2. The probabilistic response of the nonlinear random vibration system

The nonlinear equations of motion for random structure system can be expressed as [6]

$$\boldsymbol{M}(\boldsymbol{U})\ddot{\boldsymbol{x}}(\boldsymbol{U},t)+f(\boldsymbol{U},\boldsymbol{x},\dot{\boldsymbol{x}})=\boldsymbol{F}(\boldsymbol{U},t) \tag{1}$$

in which, $\boldsymbol{M}$ is the generalized mass matrix, $\boldsymbol{f}$, $\boldsymbol{x}$ and $\boldsymbol{F}$ are the vector of the nonlinear function, displacement and excitation forces, respectively. The superscript " $\cdot$ " represents the first derivative of t. $\boldsymbol{U}$ is the vector of generalized coordinates, including the parameters of stochastic loads and structure.

Suppose $\boldsymbol{q}$ is a function of matrix $\boldsymbol{U}$, then the two order Taylor expansion of $\boldsymbol{q}$ at the mean value $\boldsymbol{U}$ can be expressed as

$$q(\boldsymbol{U})=q(\bar{\boldsymbol{U}})+\left.\frac{\partial q}{\partial(cs\boldsymbol{U})^{T}}\right|_{\boldsymbol{U}=\bar{\boldsymbol{U}}}\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]+\frac{1}{2}\left.\frac{\partial^{2} q}{\partial(cs\boldsymbol{U})^{T2}}\right|_{\boldsymbol{U}=\bar{\boldsymbol{U}}}\left\{\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]\right\}^{[2]} \tag{2}$$

in which $\left\{\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]\right\}^{[2]}=\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]\otimes\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]$ is the second order Kronecker power of $\mathrm{d}\left[\mathrm{cs}(\boldsymbol{U})\right]$, symbol $\otimes$ represents the Kronecker product and $(\boldsymbol{q})_{p\times1}\otimes(\boldsymbol{U})_{s\times t}=\left[q_i\boldsymbol{U}\right]_{ps\times t}$. $\left[cs(\boldsymbol{U})\right]=\sum_{j=1}^{s}\left(e^{j}\otimes I_s\right)\boldsymbol{U}e^{j}$ and e_k is a s-dimensional unit vector with unit value in the kth element and zero value elsewhere. I_s denotes the $s\times s$ unit matrix.

Then, the matrices of both sides of Eq. (1) are expanded about $\boldsymbol{U}$ via Taylor series. By equating similar order terms, the zeroth-order, first-order, second-order equations corresponding to Eq. (1) can be obtained. According to the fourth-moment technique, the first four moments of the system responses are represented as

$$u_1(\boldsymbol{x})=\bar{\boldsymbol{x}}_0+\bar{\boldsymbol{x}}_2 \tag{3}$$

$$u_2(\boldsymbol{x})=\left[\frac{\partial\bar{\boldsymbol{x}}}{\partial(cs\boldsymbol{U})}\right]^{[2]}\left[\mathrm{Var}(cs\boldsymbol{U})\right] \tag{4}$$

$$u_3(\boldsymbol{x})=\left[\frac{\partial\bar{\boldsymbol{x}}}{\partial(cs\boldsymbol{U})}\right]^{[3]}\left[\mathrm{Tm}(cs\boldsymbol{U})\right] \tag{5}$$

$$u_4(\boldsymbol{x})=\left[\frac{\partial\bar{\boldsymbol{x}}}{\partial(cs\boldsymbol{U})}\right]^{[4]}\left[\mathrm{Fm}(cs\boldsymbol{U})\right] \tag{6}$$

in which $u_i(\boldsymbol{x})$ represent the first four moments of the system response. And the $\mathrm{Var}(cs\boldsymbol{U})$, $\mathrm{Tm}(cs\boldsymbol{U})$, $\mathrm{Fm}(cs\boldsymbol{U})$ are the variance, the third central moment and the fourth central moment of the basic random variables, respectively. $\bar{\boldsymbol{x}}_0$ and $\bar{\boldsymbol{x}}_2$ are the solutions of the zeroth order and second order equations corresponding to Eq. (1).

3. Definition of the limit state functions

The first passage failure of uncertain nonlinear MDOF system is defined as [6]

$$f(\boldsymbol{U},\boldsymbol{x})=|\boldsymbol{U}|-|\boldsymbol{x}| \tag{7}$$

where $\boldsymbol{U}=(U_1, U_2,\ldots, U_n)^{\mathrm{T}}$ is the threshold of system response $\boldsymbol{x}$. And $f(\boldsymbol{U}, \boldsymbol{x})$ defines the state function of the nonlinear MDOF system while $f(\boldsymbol{U}, \boldsymbol{x})=0$ represents the limit state function. The first four moments of the state function $f(\boldsymbol{U}, \boldsymbol{x})$ can be obtained as

$$\mu_{1fi} = \mathrm{E}\left[f_i\left(U_i, x_i\right)\right] = \mathrm{E}\left|U_i\right| - \mathrm{E}\left|x_i\right| \tag{8}$$

$$\mu_{2fi} = \mathrm{Var}\left[f_i\left(U_i, x_i\right)\right] = \mathrm{Var}\left|U_i\right| - \mathrm{Var}\left|x_i\right| \tag{9}$$

$$\mu_{3fi} = \mathrm{E}\left[f_i\left(U_i, x_i\right) - \overline{f}_i\left(U_i, x_i\right)\right]^3 = \begin{cases} \mu_{3U_i} - \mu_{3x_i}, \left(U_i, x_i\right) > 0 \\ \mu_{3x_i} - \mu_{3U_i}, \left(U_i, x_i\right) \le 0 \end{cases} \tag{10}$$

$$\mu_{4fi} = \mathrm{E}\left[f_i\left(U_i, x_i\right) - \overline{f}_i\left(U_i, x_i\right)\right]^4 = \mu_{4U_i} + \mu_{4x_i} + 6\mu_{2U_i}\mu_{2x_i} \tag{11}$$

4. System reliability analysis with dependent failure modes

4.1 Marginal probability estimation by saddlepoint approximation

The saddlepoint approximation is designed to calculate the cumulative distribution function (CDF), of the function Y=g (***X***). Suppose ***X*** is the vector of the basic random variables, $f\boldsymbol{x}(\boldsymbol{x})$ represents the probability density function of ***X***, and $M_{\boldsymbol{X}}(t)$ represents the moment generating function (MGF). Then, the MGF is expressed as

$$M_{\boldsymbol{X}}\left(t\right) = \int_{-\infty}^{\infty} \exp\left(t\boldsymbol{x}\right) f_{\boldsymbol{x}}\left(\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} \tag{12}$$

And the cumulative generating function (CGF) can be expressed as

$$K_{\boldsymbol{X}}\left(t\right) = \ln\left[M_{\boldsymbol{X}}\left(t\right)\right] \tag{13}$$

Suppose $\mu_{\boldsymbol{x}} = \left(\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n}\right)^{\mathrm{T}}$ denotes the mean value of system response ***x***. Then, the first order Taylor series expansion of $Y = g(\boldsymbol{U}, \boldsymbol{x})$ at the mean value $\mu_{\boldsymbol{x}}$ could be expressed as

$$Y = g\left(\mu_{\boldsymbol{x}}\right) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial x_i}\right|_{\mu_{xi}} \left(x_i - \mu_{x_i}\right) \tag{14}$$

The CGF of Eq. (14) can be obtained according to the functional properties discussed above.

$$K_Y\left(t\right) = \left(g\left(\mu_{\boldsymbol{x}}\right) - \sum_{i=1}^{n} \left.\frac{\partial g}{\partial x_i}\right|_{\mu_{xi}} \mu_{x_i} \right) t + \sum_{i=1}^{n} K_{X_i} \left(\left.\frac{\partial g}{\partial x_i}\right|_{\mu_{xi}} t \right) \tag{15}$$

According to the saddlepoint approximation theory [7], the PDF of Y is expressed as

$$f_Y\left(y\right) = \left\{ \frac{1}{2\pi K_Y''\left(t_s\right)} \right\}^{0.5} \exp\left(K_Y\left(t_s\right) - t_s y\right) \tag{16}$$

in which K_Y'' is the second-order derivative of the CGF of $Y = g(\boldsymbol{U}, \boldsymbol{x})$, t_s is the saddle point with the single saddle point equation

$$K_Y'\left(t\right) = y \tag{17}$$

in which K_Y' is the first-order derivative of the CGF of $Y = g(\boldsymbol{U}, \boldsymbol{x})$.

The CGF of $Y = g(\boldsymbol{U}, \boldsymbol{x})$ is provided according to the saddlepoint approximation theory,

$$F_Y\left(y\right) = P\left(Y \le y\right) = \Phi\left(w\right) + \varphi\left(w\right)\left(\frac{1}{w} - \frac{1}{v}\right) \tag{18}$$

Eq.(18) is an exact approximation of the CDF of limit state function Y. $\Phi(\cdot)$ and $\varphi(\cdot)$ represents the CDF and PDF of a standard normal distribution function. Symbol w and v is expressed as

$$w = sign(t_s)\left\{2\left[t_s y - K_Y(t_s)\right]\right\}^{0.5} \tag{19}$$

$$v = t_s\left[K_Y''(t_s)\right]^{0.5} \tag{20}$$

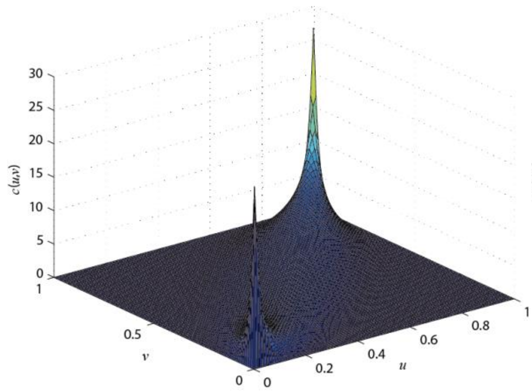
where *sign* is the sign function with sign(t_s)= 1, -1, or 0 corresponding to the cases t_s> 0, t_s < 0, or t_s = 0, respectively..

4.2 System reliability estimation using Gaussian copula

The Gaussian copula belongs to the frequently used elliptical copula family, the bivariate Gaussian copula is defined as the joint normal CDF of standard normal variables [8,9].

$$C(u,v;\theta) = \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\left(\frac{-\left(r^2+s^2-2\theta rs\right)}{2\left(1-\theta^2\right)}\right) \mathrm{d}r\mathrm{d}s \tag{21}$$

where $\Phi(\bullet)$ denotes the standard normal distribution, $\Phi^{-1}(\bullet)$ is the inverse standard normal distribution. The correlation between $u = F_1(G_1)$ and $v = F_2(G_2)$ is represented by the correlation parameter θ, which is restricted to the interval from -1 to 1. The cumulative density function and the probability density function of a Gaussian copula is plotted in figure 1.



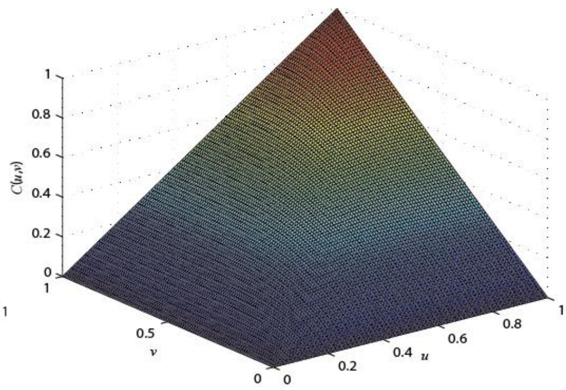


Figure 1: The PDF and CDF of a Gaussian copula.

Based on the fact in the engineering application that random variables are frequently assumed to be normally distributed, the Gaussian copula could therefore be utilized to describe overwhelming majority reliability problem [9].

To perform the joint failure probability estimation, the Gaussian copula parameter is firstly estimated with the aforementioned Kendall rank correlation coefficient, and for Gaussian copula, it is simplified as

$$\tau = \frac{2\arcsin\theta}{\pi} \tag{22}$$

With the known Kendall rank correlation coefficient, the parameter θ of the Gaussian copula can be easily estimated by equation (22). The parameter θ could indicate the degree of correlation between each pair of failure mode.

The joint failure probability could then be calculated by the Gaussian copula with the estimated parameter and gives

$$p_{ij} = C\left(u,v;\sin\left(\pi\tau/2\right)\right) \tag{23}$$

The narrow bound is obtained by calculating the failure probability of each component and joint failure probability of each pair of failure modes. With the procedure presented here, the component reliability analysis is conducted with the saddlepoint approximation. The joint probability is estimated based on the copula concept and achieved by Gaussian copula. Thus, by integrating equation (18) and (23) into the narrow bound, we propose the copula-based narrow bound as

$$P_{f1} + \sum_{i=2}^{m} \max\left(P_{fi} - \sum_{j=1}^{i-1} C\left(P_{fi}, P_{fj}\right), 0 \right) \le P_f \le \sum_{i=1}^{m} P_{fi} - \sum_{i=2}^{m} \max\left(C\left(P_{fi}, P_{fj}\right)\right) \tag{24}$$

where $C(\bullet)$ is selected as the Gaussian copula function. The system reliability could then be calculated with the copula-based narrow bound.

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