BRITISH ACOUSTICAL SOCIETY

71/19

Symposium on Finite Element Techniques in Structural Vibration
24 - 25 March 1971

The Vibration of Tower-Shaped Shells

by

H. Tottenham*, C.A. Brebbia* and J.M. Wilson**

Introduction

This paper is concerned with the prediction of the natural frequencies and dynamic response of tower-shaped structures (i.e. concrete chimneys, cooling towers, etc). The natural frequencies of a simple conical-shell chimney are calculated using finite element and matrix progression methods. For the type of concrete structure considered, damping has an important influence on the dynamic response and the determination of the physical damping properties of the concrete becomes necessary. General viscoelastic models are proposed to represent the material damping. Experimental methods of determining the damping properties of simple concrete specimens are outlined. Once the material properties are known, the problem of forced vibration due to wind, earthquakes, etc can be solved by several methods, including the step-by-step method. This method is described and discussed.

Natural Frequencies of a Chimney Structure

By using the finite element displacement formulation the equations of motion of an elastic structure reduce to the well known eigenvalue problem for free vibrations, where $\underset{\sim}{K}$ and $\underset{\sim}{M}$ are the stiffness and mass matrices.

$$\left[\underset{\sim}{K} - \omega^2 \underset{\sim}{M}\right] \underset{\sim}{U} = \underset{\sim}{0} \qquad \ldots (1)$$

The stiffness and effective 'mass' matrices can also be formed using the matrix progression method. The stiffness matrix can be exact if suitable integration procedures are used. The mass matrix is found by assuming that the difference between the static and dynamic stiffness is proportional to ω^2. This results in a formulation similar to the finite element method: the two can be combined if the same degrees of freedom are used. The modes can be found either by standard eigenvalue methods or by direct matrix progression.

A truncated conical shell approximating an actual chimney structure is used to compare the adequacy of various structural models. Three types of finite element are used to represent this cone: a uniform beam, a tapered beam and a conical shell. Using beam and shell finite element programs, the natural frequencies of the cone were computed. These are shown, together with natural frequencies obtained using a matrix progression method based on thin shell theory, in Table 1. The table shows good agreement between the finite element and matrix progression shell results and results obtained for

* Southampton University **Central Electricity Research Laboratories

a tapered Timoshenko beam. The stepped beam gives the fundamental frequency quite accurately, but the higher frequencies are more widely separated. For completeness a uniform Euler beam is used to represent the shell. Its natural frequencies, computed using finite element methods, agree well with those obtained analytically, but, apart from the fundamental, they do not correspond well with the shell theory results.

TABLE I

The Natural Frequencies (in Hz) of a Conical Shell Vibrating as a Cantilever

Cantilever Mode Number	Conical Shell		Tapered Timoshenko Beam Finite Element Method			Stepped Euler Beam Finite Element Method		Uniform Euler Beam		
								Exact	Finite	Element
	Finite Element	Matrix Progression	9	15	30	10	20		6	10
1	0.511	0.485	0.524	0.526	0.452	0.520	0.530	0.475	0.474	0.474
2	2.70	2.62	2.79	2.76	2.74	3.05	3.06	2.98	2.98	2.98
3	6.54	6.45	7.18	6.76	6.60	8.42	8.40	8.33	8.43	8.36
4	11.1	-	13.5	11.8	11.4	16.5	16.4	16.5	19.6	16.5
5	16.0	-	26.1	18.7	16.6	27.4	27.1	27.0	35.7	27.4

Height, 171 m (561 feet); upper radius, 18.0 m (59 feet); lower radius, 21.0 m (69 feet).
Thickness, 0.393 m (1.29 feet); Young's Modulus, 31,000 N/mm^2 (4.5×10^6 p.s.i.); density, 2400 kg/m^3 (150 lb./cu.ft.).
The number of degrees of freedom used is indicated at the top of the column.

The accuracy of any finite element determination of the natural frequencies and mode shapes of a structure depends on the type and number of elements used to describe the structure. For example, in the case of the uniform beam, the lower natural frequencies can be computed quite accurately using only a few degrees of freedom. However, the higher frequencies compare badly with the exact ones. By increasing the number of degrees of freedom, the accuracy of the higher frequencies can be improved.

Damping Properties of Concrete

Two hypotheses can be made about the material behaviour of concrete if the level of compressive stress is not too great. These are that concrete behaves as a linear viscoelastic material and that its Poisson's ratio can be considered as constant with time. These hypotheses, together with the assumption that concrete is a homogeneous isotropic material, allow its characterisation by a simple visco-elastic model. In principle the model parameters can be determined by a uniaxial creep test but, because the model time constants are so small, a dynamic test must be carried out in practice.

We will here propose a simple general model for conrete which utilises a combination of linear Kelvin and Maxwell elements. In most practical cases the number of Maxwell elements required to represent the material behaviour will be 1 or 2. If a steady state sinusoidal stress of frequency ω is applied to unit volume of the material, then the corresponding strain lags by the phase angle ϕ where:

$$\tan \phi = \frac{\omega \left[C_o + \sum_{k=1}^{p} \frac{C_k}{1 + \omega^2 T_k^2} \right]}{E_o + \omega^2 \sum_{k=1}^{p} \frac{C_k T_k}{1 + \omega^2 T_k^2}} \quad \ldots (2)$$

and

$$T_k = C_k / E_k \quad \ldots (3)$$

E_o, E_k, C_o and C_k are viscoelastic model parameters.

and in particular E_0 is the Young's Modulus of the material. At a resonant frequency of a structure, if the damping is small and purely viscoelastic, it is possible to write

$$\delta = \pi \tan \phi$$

where δ is the logarithmic decrement.

By measuring the variation in damping with frequency of a concrete specimen (in the form of a δ-ω curve for a structure or a ϕ-ω curve for a material sample) it is possible to deduce the viscoelastic model parameters C_o, C_k and E_k.

The δ-ω curve is obtained by exciting a structure at its natural frequencies and measuring the logarithmic decrement by any of the recognised methods. The ϕ-ω curve can be obtained using a programmed testing machine. A prismatic specimen of concrete is subjected to a sinusoidal forcing function and the strain output phase angle is compared with it.

Once the viscoelastic material model parameters are known, we can use them to find the relaxation modulus G(t) or the creep compliance J(t) of the material. These are used in the hereditary integrals relating stresses and strains.

$$\sigma(t) = \int_o^t G(t-\tau) \frac{\partial \varepsilon}{\partial \tau} d \qquad \ldots (4)$$

Because of our suppositions of constant Poisson's ratio and an isotropic material, these integrals can be extended to three-dimensional material behaviour.

$$\underset{\sim}{\sigma}(t) = \int_o^t G(t-\tau) \, \underset{\sim}{\mu} \frac{\partial \underset{\sim}{\varepsilon} (\tau)}{\partial \tau} d\tau \qquad \ldots (5)$$

where $\underset{\sim}{\mu}$ is a matrix function of Poisson's ratio and $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$ are stress and strain vectors.

The well-known Kelvin model is the simplest material representation from the structural viewpoint. This model is equivalent to assuming that the damping matrix is a linear function of the stiffness matrix.

$$\sigma = E_o \, \underset{\sim}{\mu} \, \underset{\sim}{\varepsilon} + C_o \, \underset{\sim}{\mu} \, \dot{\underset{\sim}{\varepsilon}} \qquad \ldots (6)$$

For this case the modal superposition method is valid. In general equation (6) does not reproduce the viscoelastic behaviour of concrete correctly and a more complex model has to be used. For these models the step-by-step analysis, or a technique which allows us to work in the complex plane, has to be used.

Step-by-Step Analysis

In the step-by-step method of integration the equations of motion of a structure are solved by arranging them such that the acceleration vector at the end of a time step is expressed in terms vectors known at the beginning of the time step. The time step of the integration method should be chosen to give stability as well as adequate accuracy. All modes whose natural periods are less than or close to the length of the time step will be suppressed. In this way control over accuracy is obtained.

With the presence of linear viscoelastic material damping, it becomes necessary to assume a form for the variation of acceleration with time. To test the stability and accuracy of various integration methods a uniform elastic cantilever beam subject to a transient end load was used as a test case, for which there is available an exact solution. The equations of motion were established using both finite difference and finite element methods. Among the acceleration variations taken, the most stable was obtained using $\beta = \frac{1}{4}$ in the Newmark β method [1].

For linear viscoelastic material behaviour, using the Newmark method to perform the integration, the equations of motion can be written in the form of a recurrance law and a complete record of the strain history of the material is not required. The constitutive equation of a linear viscoelastic material model can be expressed as:

$$\sigma = E_o \varepsilon + C_o \frac{\partial \varepsilon}{\partial t} + \sum_k E_k \int_o^t e^{-(t-\tau)/T_k} \frac{\partial \varepsilon}{\partial \tau} \, d\tau \quad \ldots (7)$$

The equations of motion can be written, using (7) and the finite element displacement formulation, as:

$$\underset{\sim}{M} \, \ddot{\underset{\sim}{U}}_t = \underset{\sim}{F}_e (t) - \underset{\sim}{F}_i \left(\ddot{\underset{\sim}{U}}_t \; \ddot{\underset{\sim}{U}}_{t-h} \; \dot{\underset{\sim}{U}}_{t-h} \; \underset{\sim}{U}_{t-h} \; \underset{\sim}{Q}_{k,\, t-h}\right) \quad \ldots (8)$$

where $\underset{\sim}{F}_e (t)$ is an external force vector.

$\underset{\sim}{F}_i$ is an internal force vector.

$\ddot{\underset{\sim}{U}}_t$ and $\ddot{\underset{\sim}{U}}_{t-h}$ are acceleration vectors at times t and t-h.

$\dot{\underset{\sim}{U}}_{t-h}$, $\underset{\sim}{U}_{t-h}$ and $\underset{\sim}{Q}_{k},_{t-h}$ are velocity, displacement and viscoelastic vectors respectively.

The first step in the computer algorithm is to solve equations (8) for $\ddot{\underset{\sim}{U}}_t$ with t-h and initial values of velocity, displacement vectors etc. at time t=o. The new set of vectors are then computed for time h and equations (8) are solved for t=t+h. This process is repeated for the duration of the excitation.

Conclusions

By representing concrete as a linear viscoelastic material and by using a suitable structural idealisation, the dynamic response of tower-shaped structures can be computed by a step-by-step method which is both stable and accurate.

References

[1] Newmark, N.M., July 1959, 85, EM 3, Proc. ASCE, J. of the Eng. Mech. Div., A Method of Computation for Structural Dynamics.

Acknowledgement

Part of this work was carried out at the Central Electricity Research Laboratories and the paper is published by permission of the Central Electricity Generating Board.