OF FINITE AMPLITUDE PLANE WAVES AND OF ENDFIRE ARRAYS

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This paper is divided into two quite unrelated parts. In Part I, we will be considering the propagation of the fundamental frequency component of a plane wave. And in Part II our attention will be directed towards the possibility of improving the converation efficiency of the endfire array.

PART I

PLANE WAVES OF FINITE AMPLITUDE

The propagation of finite amplitude plane waves has already been described quite thoroughly by Fay[1], Fubini-Ghiron[2], Blackstock[3] and many others. The accuracy of these findings is not in question. The aim of this study was to develop expressions which describe the propagation of the fundamental frequency component of a plane wave accurately but simply. There is some merit in having expressions requiring no more than a slide-rule for their evaluation. While the detailed approach described herein is not a fully satisfying one, the result obtained does seem to be remarkably accurate.

We will assume that we may calculate the rate at which energy is lost by the fundamental component of a wave merely by considering viscous losses and the pressure-volume work done by the fundamental component on the second harmonic. Other interactions will be neglected. I believe that Westervelt described a method of this sort at the International Congress on Acoustics held at Stuttgart in 1959, but, unfortunately, I have not yet seen a copy of this paper. The intensity of the fundamental component of a plane wave will be governed by the following equation:

$$\frac{dI_1}{dx} = -2\alpha I_1 - \overline{\frac{1}{V_1} p_2 \frac{dV_1}{dt}} \qquad (1)$$

where I_1 is the intensity of the fundamental, x is the spatial coordinate, α is the viscous absorption coefficient at the fundamental frequency ω, p_2 is the instantaneous second harmonic pressure at x, V_1 is the volume of a fluid element of unit mass at x, and where the bar indicates a time average over one period.

Several authors[4,5,6] give the following approximate expression for p_2:

$$p_2 = P_o^2 e^{-2\alpha x} \frac{\beta\,\omega}{2\,\rho_o c_o^3} \frac{(1-e^{-2\alpha x})}{2\alpha} \sin\,[2(kx-\omega t)] \qquad (2)$$

where P_o is the initial pressure amplitude (at x=0), β is a parameter of nonlinearity (equal to $\frac{1}{2}(\gamma+1)$ for gases or $(1+B/2A)$ for liquids), ρ_o is the fluid density, c_o is the sound velocity when the disturbance is infinitesmal, and k is the wave number, ω/c_o. The accuracy of equation (2) will be improved if we replace $P_o^2 e^{-2\alpha x}$ by P_1^2, the square of the fundamental amplitude at x. Indeed equation (2) modified in this way reduces to the Fubini solution near the source, and to the Fay solution in the shock wave region. At ranges much greater than α^{-1} the modified equation underestimates p_2 by a factor of 2 relative to the Fay solution [though this difficulty could be overcome by replacing $(1-e^{-2\alpha x})/2$ by $(1-e^{-2\alpha x})^2/(1-e^{-4\alpha x})$].

$\frac{1}{V_1}\frac{dV_1}{dt}$ is given by $-K_s p_1 + q$ where K_s is the adiabatic fluid compressibility and $q = (1/\rho_o^2 c_o^4)\frac{\partial}{\partial t}(p_1^2)$, is the source function developed by Lighthill[7] and simplified by Westervelt[8].

Making these substitutions and noting that the product of p_1 and p_2 averages to zero, equation (1) becomes

$$\frac{dI_1}{dx} = -2\alpha I_1 - \frac{\beta^2\omega^2}{\rho_o c_o^5} \frac{(1-e^{-2\alpha x})}{2\alpha} I_1^2 \qquad (3)$$

This is a form of Bernoulli's equation and, consulting a text on elementary differential equations, we should find the exact solution to be :

$$I_1 = I_o e^{-2\alpha x} [1 + \frac{\beta^2 \omega^2 I_o}{8 \rho_o c_o^5 \alpha^2} (1-e^{-2\alpha x})^2]^{-1} \qquad (4a)$$

or, in Blackstock's notation :

$$I_1 = I_o e^{-2\frac{\sigma}{\Gamma}} [1 + \frac{\Gamma^2}{16} (1-e^{-2\frac{\sigma}{\Gamma}})^2]^{-1} \qquad (4b)$$

where Γ is the acoustic Reynolds number and $\sigma = \Gamma\alpha x$.

Equation (4b) was checked against Blackstock's solution to Burgers' equation. It proved to be very difficult indeed to distinguish between the two solutions on Blackstocks published curves of extra attenuation. Compared with the numerical values given by Blackstock for very large ranges, the maximum discrepancy appeared to be 0.35 dB.

The results of an experiment at 8.75 MHz are compared with equation (4) in Figure 1. We see that the plane wave theory agrees with the experimental points up to a range of about 40 cm. This seems reasonable considering that the nearfield limit for the 1 cm.square transducer (Rayleigh distance) is about 60 cm. The curve marked "spreading wave" is the result of an attempt to apply this method to the sound field of a real transducer.

A second experiment (at a different frequency) was intended to determine if the 0.35 dB discrepancy noted above could be resolved experimentally. The experiment was carefully designed so that measurements could be made to a relative accuracy of 0.1 dB.

In Figure 2 we see that the experimental points fit the curves predicted by both theories with about the same degree of precision. The two sets of experimental points represent the same data shifted relative to one another along lines parallel to the diagonal representing linear propagation. Such a shift must be permitted to allow for uncertainty in the absolute values of β and I_o. We would have to know these two quantities to an accuracy better than about 2% in order to say with confidence which theory gives the better fit.

A number of useful relations may be derived from equations (3) and (4).

The extra attenuation is simply:

$$\begin{aligned} EXDB &= 10 \log_{10} \left[1 + \frac{\Gamma^2}{16} (1 - e^{-2\frac{\sigma}{\Gamma}})^2 \right] \\ &\doteq 10 \log_{10} \left[1 + \frac{\sigma^2}{4} \right], \quad \sigma \ll \Gamma \qquad (5) \\ &\doteq 10 \log_{10} \left[1 + \frac{\Gamma^2}{16} \right], \quad \Gamma \ll \sigma . \end{aligned}$$

The maximum fundamental intensity at some range x is as given by the Fay solution:

$$\begin{aligned} I_{1max} &= \frac{2 \rho_o c_o^5 \alpha^2}{\beta^2 \omega^2 \sinh^2(\alpha x)} \\ &= \frac{4 I_o}{\Gamma^2 \sinh^2 (\sigma / \Gamma)} \qquad (6) \end{aligned}$$

The attenuation coefficient for the fundamental is given by

$$\begin{aligned} \alpha_{fund} &= \alpha \left[1 + \frac{\beta^2 \omega^2}{\rho_o c_o^5 \, 2} (1 - e^{-2\alpha x}) \; I_1 \right] \\ &= \alpha \left[1 + \frac{\Gamma^2}{2} \frac{I_1}{I_o} (1 - e^{-2\frac{\sigma}{\Gamma}}) \right] \qquad (7) \\ &\doteq \alpha \left[1 + \frac{\beta \omega P_1}{2 \rho_o c_o^3 \alpha} \right], \alpha x \ll 1. \end{aligned}$$

This last approximate expression agrees with Rudnick[9] when P_1 is very small or very large, but the transition between the two extremes is slightly different.

The space-averaged attenuation coefficient $\overline{\alpha_{fund}}$ defined by the relation

$$P_1(x)=P_o e^{-\overline{\alpha_{fund}}\,x} ,$$

is

$$\overline{\alpha_{fund}} = \alpha \left\{ 1+\frac{1}{2\alpha x} \ln \left[1+ \frac{\beta^2\omega^2 I_o}{8\ \rho_o c_o^5\ \alpha^2} (1-e^{-2\alpha x})\right]\right\} \quad (8)$$

Differntiating with respect to P_o, and finding the maximum slope, we obtain

$$\left.\frac{d\ \overline{\alpha_{fund}}}{dP_o}\right|_{max} = \frac{\pi}{4}\ \frac{(1-e^{-2\alpha x})}{2\alpha x} \left[\frac{2\beta\ f}{\rho_o c_o^3}\right] \quad (9)$$

where $f = \omega/2\pi$. I mention this quantity because it seems to have been assumed in the literature[10] that this quantity is given by the term in square brackets alone. It also occurs to me that measurements of $\overline{\alpha_{fund}}$ might well yield accurate values of the parameter β.

In closing, I would like to say that we may also use the expression for the second harmonic developed by Pernet[6] and Safar[11] to obtain a similar solution for spherical waves. But unfortunately we must abandon our slide rule if α is finite.

PART II

The Efficiency of the Endfire Array

The generation of sum and difference frequencies by two strong interfering sound waves has been a subject of discussion for some hundreds of years. Helmholtz[12] credits the original observations to Sorge and Tartini in the period 1740-1750. Since then the subject has received the attention of several authors, but until the last ten years or so, the effect seems to have been regarded either as just a spurious, undesirable nuisance, or as a rather academic subject. With thought now directed towards applications, it seems appropriate to question whether or not the 'traditional' treatment of the interaction between two sound waves leads to the most efficient scheme for generating an interaction component at a frequency below that of the transmitted wave or waves. We shall adopt for our analysis the quasi-linear, source function approach due to Rayleigh[13], Lighthill[7], Westervelt[8], and Berktay[14]. Berktay considers a primary wave of the form

$$p(t)=E(t)\ \sin\ (\omega t+\phi) \qquad (10)$$

where E(t) represents the envelope of the pressure wave and ω is the primary carrier or center frequency. If E(t) has no components higher in frequency than $\omega/3$, then there will be no overlap in the frequency spectra of the scattered and primary waves. Berktay shows that, taking frequencies up to $2\omega/3$, the farfield pressure waveform will be of the form

$$p_s(t)=\text{const.}\ \frac{\partial^2}{\partial_t^{\ 2}}\ E^2(t')$$

where t' is the retarded time $t-R/c_o$.

In this analysis we will take $E(t)$ to be a periodic function having period T or repetition frequency $\Omega=2\pi/T$, and will confine our attention to the scattered component at frequency Ω. The magnitude of this frequency component will be proportional to the quantity

$$\Omega^2 \int_0^T E^2(t) \cos \Omega(t-\tau) \, dt$$

where τ is adjusted to give a maximum. The factor Ω^2 outside the integral accounts for the double differentiation with respect to T.

We then define a figure of merit G:

$$G = \frac{\Omega^2 \int_0^T E^2(t) \cos \Omega(t-\tau) \, dt}{\int_0^T E^2(t) \, dt} \tag{11}$$

G is a measure of the efficiency of the nonlinear conversion process in that G is proportional to ratio of the amplitude of the scattered signal at a frequency Ω, to the average power transmitted at the primary frequency.

We would like to optimize G by shaping $E^2(t)$, while at the same time holding $\int_0^T E^2(t)dt$ constant. Now the cosine term in equation (11) acts as a weighting function having extreme values of ± 1. We will maximize G, then, by concentrating $E^2(t)$ as much as possible at one of these extreme values. In other words, we would like to represent $E^2(t)$ by $\delta(t-\tau)$. We then obtain

$$G_{opt} = \Omega^2 \tag{12}$$

Let us now compare this optimum figure with that obtained for the 'conventional' two-frequency type transmission. If the two primary frequency components have equal magnitudes we may let $E(t)=\cos(\Omega t/2)$

so that $E^2(t)=\cos^2(\Omega t/2)=\frac{1}{2}(\cos(\Omega t)+1)$.

We obtain

$$G_{conv}=\Omega^2 \frac{\frac{1}{2}\int_0^T \cos^2(\Omega t)\, dt + 0}{\int_0^T \cos^2(\Omega t/2)\, dt}$$

$$= \frac{\Omega^2}{2} \cdot \qquad (13)$$

Equation (13) tells us that the 'conventional' system is considerably suboptimum. One could, in principle, obtain four times as much power at the 'difference' frequency by transmitting the same average primary power in a different manner.

Of course, the optimum system- modulating the carrier by the square root of a delta function - cannot be realized. And in any case, our assumption that $E(t)$ contains only frequency components below $\omega/3$ is violated. Let us therefore consider something a little closer to reality. Consider that we represent $E^2(t)$ by a train of rectangular pulses of width wT as shown in Figure 3. We let

$$E^2(t) = \frac{1}{2w} \qquad -wT/2 < t < wT/2, \quad 0 < w < 1$$

$$= 0 \qquad \text{otherwise}$$

Then

$$G_w = \frac{2\Omega^2}{wT} \int_0^{wT} \cos\Omega t\, dt$$

$$= \Omega^2 \frac{\sin(\pi w)}{\pi w} \qquad (14)$$

This result tells us that we should try to make w as small as possible. However, at $w=0.25$ - quite a realistic sort of value - $G=0.9$ so that we are still better off than the 'conventional' system by 5.1 dB. The performance of the pulse modulation system is described in Figure 4. This figure has been normalized so that the performance of the two-frequency system is described by the horizontal line at 0 dB.

The case $w=\frac{1}{2}$ is of particular significance. When $w=\frac{1}{2}$, the peak envelope power is twice the average power just as in the 'conventional' case. Yet the scattered component at frequency Ω is 2.1 dB greater for the pulsed transmission. If the constraint imposed by the equipment is in terms of peak envelope power rather than average power, it may be shown that $w=\frac{1}{2}$ signifies the optimum mode of transmission.

An experimental comparison has been carried out in water, operating an endfire array in the 'conventional' mode and in the $w=\frac{1}{2}$ pulsed mode. The carrier or center frequency used was 8.75MHz. The difference or pulse repetition frequencies used were 100kHz, 150kHz, and 300kHz. The same average and peak envelope powers were transmitted in each case. The pulsed system was found to yield a scattered component 2.1 ± 0.4 dB greater than two frequency system.

The pulsed transmission will also yield scattered frequency components which are harmonics of the repetition frequency. The structure of the scattered wave frequency spectrum will, of course, depend upon w and upon the bandpass characteristics of the primary transmitter. Generally, the first one or two harmonics - perhaps more - will be comparable in magnitude to the fundamental component. These signals are a "bonus" which may or may not be useful.

The use of a rectangular envelope will also allow simplification and higher efficiency on the electronic side of the transmitter. The power amplifier, for example, may then be of the switching mode (class D) variety. The signal generating circuits may also be of the binary type.

The biggest limitation placed on the use of pulses, is the requirement for a greater primary-frequency bandwidth. The bandwidth requirement may be reduced by a factor of two if the carrier is modulated in both amplitude and phase[15], though the electronic complications are then considerable. The transducer bandwidth requirement is, however, a technological problem rather than a fundamental acoustic one.

A more fundamental consideration must be the sort of thing discussed in Part I - high intensity attenuation of the primary wave. Mellen, Konrad, and Browning[16] have shown that at very high intensities the scatter signal becomes linear rather than square law. Indeed it may be shown that under such conditions,

$$p_s(t) = \text{const.} \frac{\partial^2}{\partial_t{}^2} \left| E(t') \right| .$$

For this limiting case we might define a new figure of merit G'.

$$G' = \frac{\left\{ \Omega^2 \frac{1}{T} \int_o^T \left| E(t) \right| \cos \Omega t \, dt \right\}^2}{\frac{1}{T} \int_o^T E^2(t) \, dt} \tag{15}$$

This figure of merit is now a ratio of average powers and hence is a truer measure of conversion efficiency than was G.

For rectangular pulses we find

$$G'_w = \frac{\Omega^4}{\pi^2} \frac{\sin^2 (\pi w)}{w} \tag{16}$$

$$G'_{w_{opt}} = \frac{2.4\ \Omega^4}{\pi^2} \quad \text{where } w_{opt} = 0.37, \tag{17}$$

and

$$G'_{\frac{1}{2}} = \frac{2\ \Omega^4}{\pi^2} . \tag{18}$$

For the two-frequency case, $G' = 8\,\Omega^4/9\pi^2$. Thus when very high transmitted intensities are employed, the $w = \frac{1}{2}$ pulse system will yield a scattered component 3.5 dB greater than the 'conventional' system, while the maximum improvement possible is now only 4.3 dB.

To answer our original question then, the interaction between two monochromatic sound waves does <u>not</u> lead to the most efficient generation of a scattered component lower in frequency than the primary wave or waves, unless transducer bandwidth is a serious limitation. If the necessary bandwidth is available, a pulsed carrier type of transmission will give an improvement in efficiency of between 2 and 6 dB depending upon the system constraints.

References

1. R.D. Fay, J. Acoust.Soc. Am. 3, 222-241 (1931)

2. E. Fubini-Ghiron, Alta Frequenza 4, 530 (1935)

3. D.T. Blackstock, - J. Acoust. Soc. Am. 34, 9-30 (1962)
   - J. Acoust. Soc. Am. 36, 534-542 (1964)
   - J. Acoust. Soc. Am. 39, 1019-1026 (1966)

4. Z.A. Gol'dberg, Soviet Phys.-Acoust. 3, 157-162 (1957)

5. W. Keck and R.T. Beyer, The Physics of Fluids 3, 346-352, (1960)

6. D.F. Pernet, Symposium on Aerodynamic Noise held at Loughborough University, Paper C2 (1970)

7. M.J. Lighthill, Proc. Roy. Soc. A211, 564-587 (1952)

8. P.J. Westervelt, J. Acoust. Soc. Am. 35, 535-537 (1963)

9. I. Rudnick, J. Acoust. Soc. Am. 30, 564-567 (1958)

10. V. Narasimhan and R.T. Beyer, J. Acoust. Soc. Am. 29, 532, (1957)

11. M.H. Safar, J. Sound Vib. 13, 1-7 (1970)

12. H.L.F. Helmholtz, On the Sensations of Tone, Chapter VII, (trans. by A.J. Ellis) Longmans, London, (1875)

13. J.W.S. Rayleigh, The Theory of Sound, Vol.II, p. 154, Dover, New York (1945)

14. H.O. Berktay, J. Sound Vib. 2, 439-461 (1965)

15. K.H. Powers, Proc. IRE 48, 1431-1435 (1960)

16. R.H. Mellen, W.L. Konrad, and D.C. Browning, elsewhere in these proceedings.

FIGURE CAPTIONS

Figure 1. Experiment and theory (eqn.4) compared at 8.75 MHz.

Figure 2. Comparison of eqn.(4) with Blackstock's solution to Burgers' eqn. and are experimental data fitted to the two respective theories.

Figure 3. Power envelope, $E^2(t)$, for a train of rectangular pulses.

Figure 4. Comparison of efficiency and peak power in the pulse-mode and two-frequency mode endfire arrays. These quantities are normalized so that the performance of the two-frequency array is described by unity (0 dB).

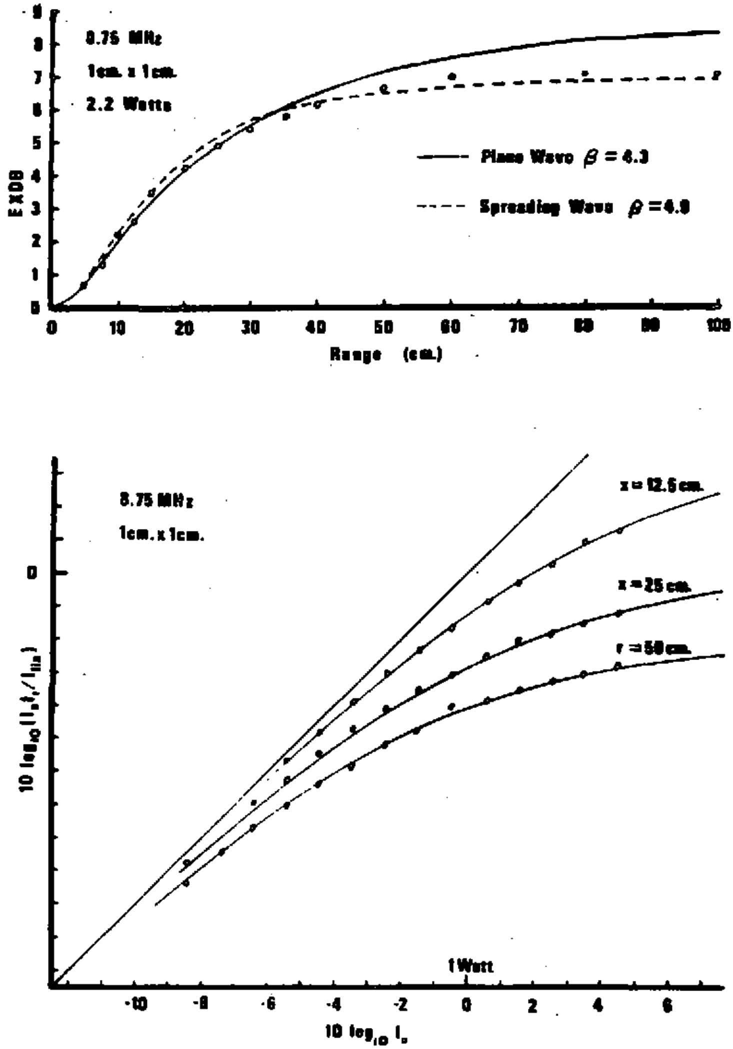
8.75 MHz
1cm. x 1cm.
2.2 Watts
Plane Wave β = 4.3
Spreading Wave β = 4.9
EXDB
Range (cm.)
8.75 MHz
1cm. x 1cm.
r = 12.5 cm.
r = 25 cm.
r = 50 cm.
1 Watt
10 log₁₀ I₀

FIGURE E

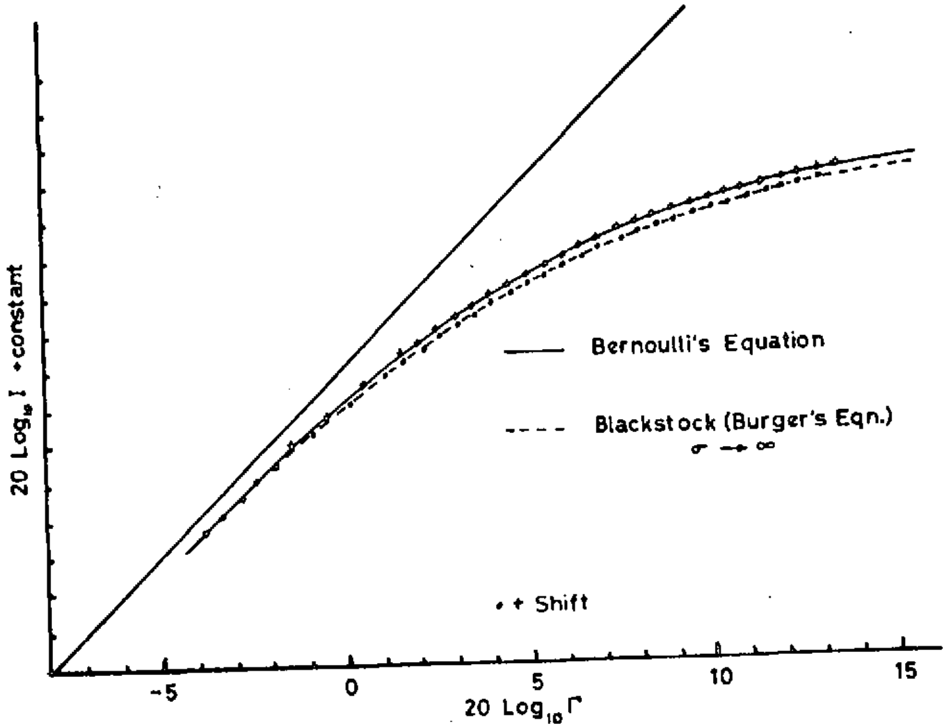
Bernoulli's Equation
Blackstock (Burger's Eqn.)
σ → ∞
• + Shift
20 Log₁₀ I + constant
-5
0
5
10
15
20 Log₁₀ Γ

FIGURE 2

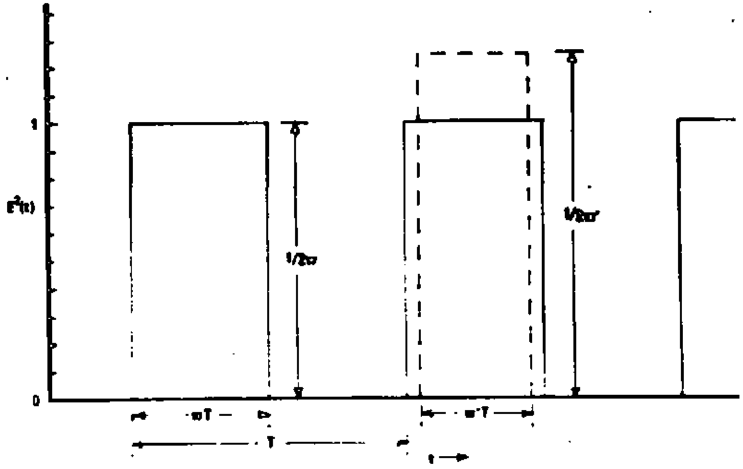

FIGURE 3

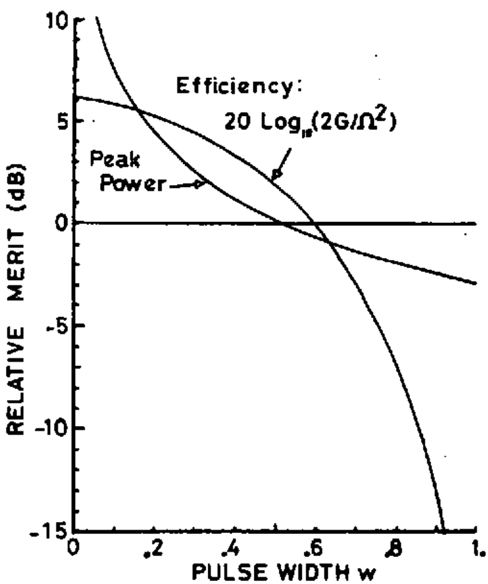

FIGURE 4