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# COMPLEX MOTIONS OF A FLUID-LOADED NONLINEAR ELASTIC PLATE; PERIODIC, SUBHARMONIC AND CHAOTIC VIBRATIONS.

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## 1. INTRODUCTION

In this paper a brief account is given of the motion of a baffled plate, immersed in a compressible fluid, and subjected to acoustical or mechanical forcing of sufficient size that a nonlinear term in the plate's governing equation cannot be ignored. We uncouple the mathematical system by allowing the fluid to be light compared to the plate, and then employ asymptotic techniques to reduce the problem to the determination of plate modal amplitudes via a solution of nonlinear differential equations. We study three particular configurations in this paper; two forced by plane incident acoustic waves, and one via mechanical vibration of the plate ends. When the incident waves are of a single frequency, close to a plate natural resonance, the nonlinear equations give simple fixed point solutions after the transients have decayed [1]. If two nearly coincident frequency waves irradiate the plate, then the equations are much more complex ((4.11), (4.13)) and give rise to harmonic, subharmonic, and aperiodic (chaotic) plate deflections. These features are illustrated in the numerical results presented herein.

The first model we present includes an in-plane compressive load which almost pushes the plate into its first static buckling mode. Vibrations of in-vacuum buckled plates or columns have been investigated by many authors (e.g. [2, 3, 4]). The usual approach is to truncate the infinite system of modes thereby reducing the problem to a system of ordinary differential equations of the forced Duffing type. Any justification for this truncation, such as the asymptotic ordering presented here is usually omitted.

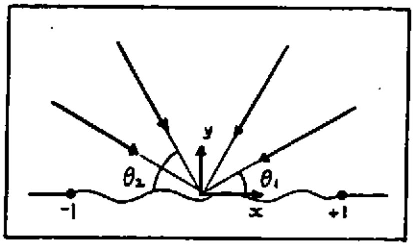


FIGURE 1.

Of particular interest is the result that the incident energy at the acoustic frequency is channelled into the slowly vibrating buckled mode, and it is this mode which satisfies a nonlinear Mathieu equation. The case of mechanical forcing also leads to this equation. Finally, with no-compressive load, only one excited resonant mode is shown to persist for long times, and this satisfies an equation whose solutions are illustrated in figures 3 and 4.

## 2. THE BOUNDARY VALUE PROBLEM

We now pose the general mathematical problem for investigation and introduce non-dimensional scalings to reduce the problem down to its simplest form. Three different physical cases will be presented for study.
We examine the motion of a thin elastic plate of width 2a, and infinite length, forced by vibrations perpendicular to the plate axis, and simply supported by a plane rigid infinite baffle. Cartesian coordinates $(\bar{x},\bar{y})$ are shown, for this two-dimensional configuration, in figure 1 and a compressible inviscid fluid occupies the half space above the plate $(\bar{y}>0)$ whereas, for simplicity, we take a vacuum below.

The (dimensional) plate deflection $\bar{v}$ is governed, for reasonably small displacements, by the equation

$$B\frac{\partial^4\bar{v}}{\partial\bar{x}^4} + \left\{\bar{\lambda} - N\left[\int_{-a}^{a}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2 d\bar{x}\right]\right\}\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + m\frac{\partial^2\bar{v}}{\partial\bar{t}^2} = -p(\bar{x},0),\ \bar{y}=0,\ |\bar{x}|<a, \qquad (2.1)$$

where $B = Eh^3/12(1-\upsilon^2)$ is the plate bending stiffness, $N = Eh/4a$, $E$ and $\upsilon$ the Young's modulus and Poisson's ratio respectively, $h$ the plate thickness, $m$ the mass/area of the plate, $\bar{t}$ is time, and $p(\bar{x},0)$ the perturbation pressure exerted on the plate by the fluid. The second term on the left hand side is due to the presence of plate tension; $\bar{\lambda}$ being a measure of the (given) in-plane compressive load, a parameter we are free to vary, and the nonlinear integral is a result of the stretching of the plate under bending deformations.
The following non-dimensional variables are taken:

$$x = \bar{x}/a,\ y = \bar{y}/a,\ t = \bar{t}\surd\{B/ma^4\},\ v = \bar{v}\surd\{Na/B\}, \qquad (2.2)$$

where $\surd\{ma^4/B\}$ is time-scale based on the plate, together with a velocity potential for the fluid $\Phi(x,y,t)$ which satisfies

$$p = -(\rho B/ma^3)\surd\{B/Na\}\frac{\partial\Phi}{\partial t}, \qquad (2.3)$$

with $\rho$ the fluid ambient density. The non-dimensional fluid propagation speed $c$ may be related to the original phase speed $\bar{c}$ by $c = \surd(ma^2/B)\bar{c}$, the in-plane load parameter is written as $\lambda = \bar{\lambda}a^2/B$, and a fluid loading parameter is introduced as

$$\epsilon = \rho a/m,$$

which is essentially a ratio of the fluid density to the plate density. This will be taken as small in all that follows.
The boundary value problem may now be written as:

$$v_{xxxx} + \lambda v_{xx} + v_{tt} - \left(\int_{-1}^{1} v_x^2 dx\right)v_{xx} = \epsilon(\phi_t + \Phi_t^i),\ y = 0,\ |x|<1, \quad (2.4)$$

$$\phi_{xx} + \phi_{yy} = (1/c^2)\phi_{tt}, \quad \text{all } x,\ y>0, \tag{2.5}$$

$$\phi_y = v_t, \quad \text{all},\ y = 0, \tag{2.6}$$

with edge conditions

$$v_{xx} = v = 0, \quad y = 0,\ |x| = 1, \tag{2.7}$$

baffle condition

$$v = 0 \quad y = 0,\ |x| > 1, \tag{2.8}$$

and the total acoustic potential is split into a given incident (and reflected) plane wave forcing term $\Phi^i$ and a scattered field $\phi(x,y,t)$. Note that subscripts denote partial derivatives. To fully specify the boundary value value problem we insist that $\phi(x,y,t)$ contains purely outgoing waves at infinity and further we seek solutions at large times, i.e. transients have decayed to zero, when only persistent motions (periodic or aperiodic) remain. Of primary interest in this investigation is the scattered potential $\phi(x,y,t)$, especially when it is not small. In general, it may be seen from (2.4) that the displacement is of order $\epsilon$ times the forcing amplitude, and so too, through (2.6), will be $\phi$. However, if the plate is forced close to an in-vacuo resonant frequency of the elastic plate, then the displacement and consequent radiated acoustic field is of the same order as the incident wave. We will examine the problem in this frequency regime, and we will restrict our attention primarily to two particular cases.

## 3. PARTICULAR EXAMPLES

### 3.1 Buckled Plate.

If the plate oscillates sinusoidally in a vacuum, and vibrations are so small that the nonlinear term is negligible, the plate displacement can be written in modal form (satisfying 2.7)

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{i\omega_n t} + \text{c.c}, \tag{3.1}$$

where $a_n$ are arbitrary coefficients, c.c. denotes the complex conjugate, and

$$\omega_n = n\pi\surd\{(n\pi)^2 - \lambda\}, \quad n = 1,2,3,\ldots\ . \tag{3.2}$$

Now, if $\lambda \approx \pi^2$ then the first resonance frequency drops to near zero and the plate deforms into its first static buckled mode. Here we take

$$\lambda = \pi^2 + \epsilon^2\eta \tag{3.3}$$

where $\eta$ is a buckling detuning parameter. Further we will take two incident plane waves (symmetric in x for mathematical convenience) together with their reflected waves, in the forcing term

 

$$\Phi^{i}(x,y,t) = \epsilon 4\upsilon_1\cos(\Omega_1 x\cos\theta_1/c)\cos(\Omega_1 y\sin\theta_1/c)e^{i\Omega_1 t}$$
$$+ \epsilon 4\upsilon_2\cos(\Omega_2 x\cos\theta_2/c)\cos(\Omega_2 y\sin\theta_2/c)e^{i\Omega_2 t} + \text{c.c.}, \qquad (3.4)$$

where $\epsilon\upsilon_1$, $\epsilon\upsilon_2$ are the two incident wave amplitudes ($\upsilon_1,\upsilon_2 = O(1)$), $\Omega_1,\Omega_2$ are the incident wave frequencies

$$\Omega_1 = \omega_p + \epsilon\sigma_1, \quad \Omega_2 = \omega_p + \epsilon\sigma_2, \qquad (3.5)$$

$\theta_1,\theta_2$ are the incident wave angles shown in figure 1 and $\omega_p$ is the pth resonant frequency of (2.10) which we henceforth just denote by $\omega$.
If we can obtain the deflection in the form (2.9) then the scattered field can be constructed from Green's theorem as

$$-\frac{1}{2}\sum_{n=1}^{\infty}\omega_n a_n e^{i\omega_n t}\int_{-1}^{1}\cos(n\pi x_1)H_0^{(2)}(\omega_n[(x-x_1)^2 + y^2]^{½}/c)dx_1 + \text{c.c.}, \quad y>0, \qquad (3.6)$$

where $H_0^{(2)}$ is a Bessel function of the third kind. Note that this term ($\phi$) acts as a damping term in (2.4), due to plate energy radiating off to infinity in the half space above the plate. Therefore, purely for clarity of exposition, we will simplify the analysis by taking the governing equation to have the form

$$v_{xxxx} + \lambda v_{xx} + \epsilon k v_t + v_{tt} - \left(\int_{-1}^{1} v_x^2 dx\right)v_{xx} = \epsilon\Phi_t^{i}$$
$$\sim 4i\epsilon^2\omega\{\upsilon_1\cos(k_1 x)e^{i\Omega_1 t} + \upsilon_2\cos(k_2 x)e^{i\Omega_2 t}\} + \text{c.c.} \qquad (3.7)$$

in which $k_1 = \omega\cos(\theta_1)/c$, $k_2 = \omega\cos(\theta_2)/c$ and k is a positive constant. This reduces the boundary value problem down to the single unknown v. The full problem will be tackled in a forthcoming paper [5].

3.2 Single Mode.
We will examine the amplitude of oscillations when the plate is not subjected to in-plane compression, i.e. $\lambda = 0$, and we will also take two nearly coincident waves as the forcing at frequencies close to $\omega$. Thus we will employ (3.7) with the second term on the left hand side omitted, and note that $v_t$ could, in all problems, be a small damping factor of the plate as well as due to radiation damping.

3.3 Mechanical Loading.
One interesting problem is that of radiation from the plate when the forcing is supplied through the plate ends. One could envisage acoustic energy, from engines etc., vibrating the pin-jointed edges in a sinusoidal motion along the direction of the plate (i.e. along y = 0). This would alter the in-plane tension slightly so that we may have

$$\lambda = \pi^2 + d\epsilon\cos(\zeta t) + \epsilon^2\eta \qquad (3.8)$$

where d,ζ are the vibration amplitude and angular frequency respectively. We take $\Phi^1(x,y,t)$ to be zero in this example.
These three cases will be tackled in the next section by the method of two-timing, and the problems will be reduced to ordinary differential equations for the modal coefficients.

## 4. ASYMPTOTIC SOLUTION

### 4.1 Buckled Plate

We will now attempt to determine the solution of the simplified model equation (3.7). As the forcing frequencies $(\Omega_1,\Omega_2)$ are close to $\omega$ we can expect the plate to vibrate at this frequency $\omega$ but with a slow modulation in time. To account for this slow variation we introduce a slow time variable $\tau$, defined as

$$\tau = \epsilon t \qquad (4.1)$$

which we treat as a new independent variable, so that

$$\partial/\partial t \to \partial/\partial t + \epsilon\partial/\partial\tau,\ \partial^2/\partial t^2 \to \partial^2/\partial t^2 + 2\epsilon\partial^2/\partial t\partial\tau + \epsilon^2\partial^2/\partial\tau^2. \qquad (4.2)$$

Also, as the plate is compressed so that it is nearly in its first buckled mode, we can expect some of the incident wave energy to be transferred into this lowest mode (which oscillates very slowly on the time scale $\tau$). With this in mind we propose an expansion for the displacement as

$$v = \epsilon(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots), \qquad (4.3)$$

which, on equating terms of $O(\epsilon)$, gives to leading order

$$v_{0xxxx} + \pi^2 v_{0xx} + v_{0tt} = 0. \qquad (4.4)$$

This has a solution in the modal form of (3.1) if we satisfy the jointed edge conditions (2.7), but because of the buckling ($\omega_1 = 0$) we write it as

$$v_0 = b\ \sin\pi x + \left\{\sum_{n=2}^{\infty} A_n \sin n\pi x\ e^{i\omega_n t} + \text{c.c.}\right\}. \qquad (4.5)$$

To simplify the analysis we now make the assumption that all the coefficients $A_n$, $n \neq p$, tend to zero for large times as they are not forced by the incident waves. Strictly we should keep these terms and prove later that this is the case. Thus we just take

$$v_0 = b(\tau)\sin\pi x + (A(\tau)e^{i\omega t} + \bar{A}(\tau)e^{-i\omega t})\sin p\pi x \qquad (4.6)$$

where A is the coefficient of the pth mode, $\bar{A}$ is the complex conjugate of A and both b and A are assumed to be as yet unknown functions of $\tau$. To solve for b and A we examine the $o(\epsilon^2)$ problem of (3.7), namely

$$v_{1xxxx} + \pi^2 v_{1xx} + v_{1tt} = -kv_{0t} - 2v_{0t\tau} + 4i\omega[\upsilon_1 \cos(k_1 x)e^{i(\omega t+\sigma_1\tau)} + \upsilon_2\cos(k_2 x)e^{i(\omega t+\sigma_2\tau)} + c.c.]. \tag{4.7}$$

If the asymptotic ordering in (4.3) holds for all time then $v_1$ must not contain growing terms in t. This gives a condition on the right hand side of (4.7) (the so-called secularity condition) as

$$-(kA + 2A') + 4(f_1 e^{i\sigma_1\tau} + f_2 e^{i\sigma_2\tau}) = 0 \tag{4.8}$$

where the prime denotes $d/d\tau$, $f_1$, $f_2$ are now just constants related to the forcing amplitudes, and the solution is

$$A = \frac{4f_1}{2i\sigma_1 + 1} e^{i\sigma_1\tau} + \frac{4f_2}{(2i\sigma_2 + 1)} e^{i\sigma_2\tau}. \tag{4.9}$$

Note that the complementary function of (4.8) decays exponentially with time and so is removed. This gives us half of the leading solution to v, but the buckling amplitude b has not yet been found. To obtain this we look at the next order in our expanded equation ($\epsilon^3$), and eliminate the secular terms by setting

$$-\eta\pi^2 b + kb' + b'' + \pi^4 b(b^3 + 2p^2 A\bar{A}) = 0. \tag{4.10}$$

Substituting (4.8) and its conjugate into this equation, and rearranging, leads to the nonlinear Mathieu equation:

$$\frac{d^2b}{d\tau^2} + k\frac{db}{d\tau} + (a_1 + a_2\cos[(\sigma_1-\sigma_2)\tau])b + \pi^4 b^3 = 0. \tag{4.11}$$

Here $a_1$ can be altered by adjusting the buckling detuning parameter $\eta$, $a_2$ by altering the incident wave amplitudes, and $(\sigma_1 - \sigma_2)$ is varied as the incident frequency is altered. Note that the crucial nonlinear term $b^3$ comes from the integral term in (3.7). Therefore to obtain the deflection v (and scattered field via (3.6)) we need to determine a numerical solution for (4.11).

4.2 Single Mode.

If the in-plane compression is absent, then, with the incident wave of $O(\epsilon)$ as in the above example, the displacement $\sim v_0$ is given by (4.6) with $b = 0$. Therefore the nonlinearity has not been brought into play and the solution is particularly simple. However, if we increase the incident wave amplitude to $O(\epsilon^{1/2})$ (in (3.4)) then we find

$$v \sim \epsilon^{1/2}(v_0 + \epsilon v_1 + \ldots). \tag{4.12}$$

With an expansion of (3.7), using (4.12) and the two-timing derivatives (4.2), the coefficients of $A_n$, $n \neq p$, are again shown to tend to zero at large time.

However now the complex function $A_p = A$ may be shown to satisfy

$$\frac{dA}{d\tau} + a_1 A + i a_2 A^2 \bar{A} + a_3 \cos(\tau) = 0 \qquad (4.13)$$

where, for simplicity, we have taken $f_1 = f_2$ in (4.9), $\sigma_1 = -\sigma_2$, and $\tau$ has been rescaled on $\sigma_1$. The coefficients $a_1$; $a_2$; $a_3$ are now related to the detuning frequency parameter $\sigma_1$ and the damping k; the resonance number p; and the forcing amplitude $f_1$ respectively. Thus we have a good deal of freedom to choose their numerical values. As will be seen, these values greatly influence the nature of the deflections.

### 4.3 Mechanical Loading

If $\zeta$ in (3.8) were taken to be small (i.e. slow vibrations) then we could show that the leading order displacement $v_0$ just contained one term, the buckling mode. Substituting this into the $O(\epsilon^2)$ term in the expansion of (3.7), and again eliminating the secular parts, leads to an equation for b which is identical in form to that written in (4.11). Thus the solution of this equation contributes to both physical models.

## 5. DISCUSSION OF CHAOTIC AND PERIODIC SOLUTIONS

### 5.1 Buckled Plate

The governing equation for the buckling mode b (4.11) reduces to Duffing's equation if $\sigma_1 = \sigma_2$. This equation permits periodic solutions and if perturbed slightly, i.e. $a_2 = 0$, then these periodic solutions remain. However, if $a_2$ is large enough then, as for the well studied forced Duffing's equation (see e.g. Guckenheimer & Holmes [4]), the unstable and stable manifolds of the saddle at the origin in the b, b' plane cross transversely. This leads to a homoclinic explosion and the onset of chaotic motion. Two examples for the parameter values $a_1 = -1.$, $k = .1$, $\sigma_1 - \sigma_2 = 1.$, (with b scaled so that the $b^3$ term has coefficient 1) are given if figure 2, the curves representing the point (b,b') plotted in time. A detailed analysis of this work will be presented in a forthcoming paper [5]. Note that the onset of chaos at $a_2 \sim .3375$ is preceeded by periodic orbits of high period as illustrated in fugure 2a, and the aperiodic trajectories have been plotted long after the start of the motion.

### 5.2 Single Mode

Figures 3 and 4 (except for 4a) illustrate Poincaré plots of the motion, in which only the points at times $t = 2n\pi$, n = integer, are plotted on the Complex A plane. In figure 3 the coefficients are chosen as $a_1 = .1$, $a_2 = 2.$, and the forcing amplitude $a_3$ is varied to illustrate the very different solutions that can be obtained. For $a_3$ less than .5 the Poincaré plot shows just one point, which obviously corresponds to a periodic solution of A, with period $2\pi$. As the parameter is increased the motion becomes chaotic, but still shows some order. For instance 3b illustrates that the trajectory mostly cuts the plane in two distinct regions, thereby giving an oscillating solution roughly similar to a period two orbit (period $4\pi$) (see also figure 4c). As the parameter $a_3$ further increases, the chaos sometimes dramatically collapses to simple periodic orbits, as illustrated for the $6\pi$ period orbit in figure 3c (and $2\pi$ periodic orbit at $a_3 = 8.$) but returns just as suddenly.

A more logical (but less physically meaningful) way of examining the onset of chaos is to vary the damping term $a_1$. Figure 4 demonstrates this for decreasing damping, and remaining parameter values set at $a_2 = 3.$, $a_3 = 5.$ Figure 4a illustrates the whole trajectory when $a_1 = .5$ starting from (3.,3.) and clearly indicates the attracting nature of the chaotic motions. Indeed all trajectories, for all parameter values, are attracted into a finite region around the origin. The attractor is nearly periodic in 4a, but is folded to produce wandering motions, with slightly varying periodicity in time and space. The folding of the attractor is illustrated in the Poincaré plot of 4b, and note that if $a_1 = 6.$, the attracting orbit becomes a simple limit cycle. For a damping factor of $a_1 = .12$, the motion, although chaotic, behaves quite similarly to a period three orbit, but as the damping diminishes the apparant order becomes less easy to visualize. As a striking example, for $a_1 = .01$, figure 4d gives apparantly very random motions lying within a subset of an elliptical area.

## 6. CONCLUSIONS

This paper has illustrated how the coupled motions of a fluid loaded elastic plate may be analysed when the loading parameter $\epsilon$ is small. For certain incident wave, or mechanical forcings the plate motion may be described to leading asymptotic order in terms of the amplitudes of particular modes of vibration. These amplitudes are found by the numerical solution of differential equations, which describe the slow time variations of the functions. Of primary concern is the sensitivity of the motions to particular values of the parameters, and this is clearly illustrated in figures 2, 3 and 4. In all examples the chaotic oscillations of the plate result in a chaotically varying scattered wave amplitude in the fluid. Note that for the buckling case, these oscillations occur at very low frequencies, but for the single mode case the primary resonant mode has a chaotic amplitude. Further aspects and details of this work will be presented in further publications [5].

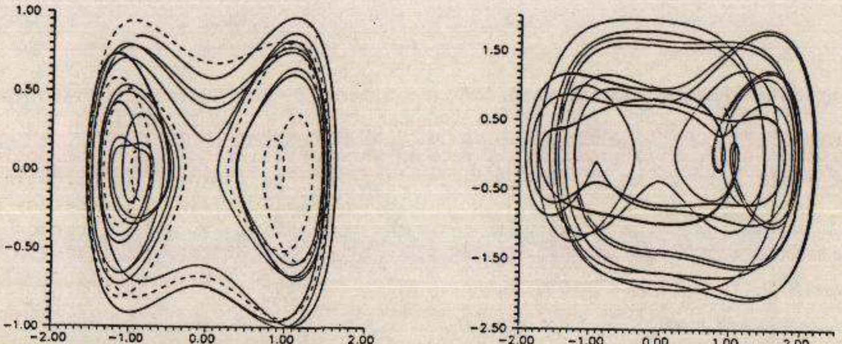


FIGURE 2. db/dτ against b for (4.11) with $a_1$ = -1., k = .1, $\sigma_1 - \sigma_2$ = 1. and a) $a_1$ = .3375 (dashed line .3374), b) $a_1$ = 1.5.

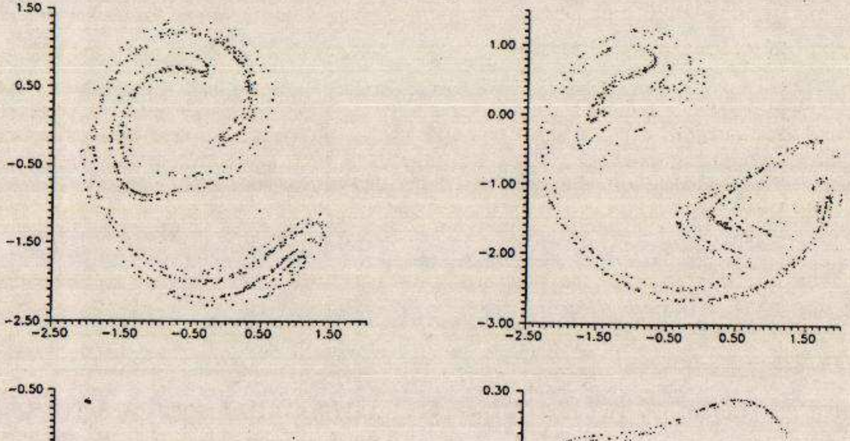


FIGURE 3. Poincaré plots of complex A for (4.13) with $a_1$ = .1, $a_2$ = 2. and a) $a_3$ = 1., b) $a_3$ = 2., c) $a_3$ = 6., d) $a_3$ = 7.

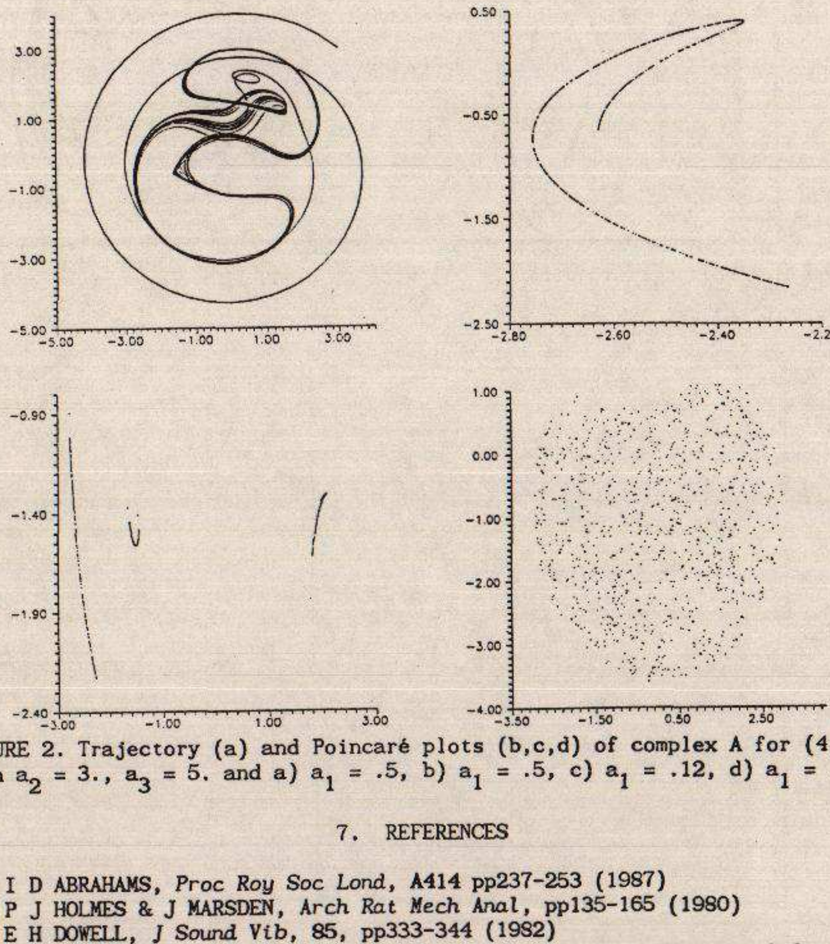


FIGURE 2. Trajectory (a) and Poincaré plots (b,c,d) of complex A for (4.13) with $a_2$ = 3., $a_3$ = 5. and a) $a_1$ = .5, b) $a_1$ = .5, c) $a_1$ = .12, d) $a_1$ = .01.

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