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THE SCATTERING OF ULTRASOUND BY A SMALL SURFACE IMPERFECTION

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1. INTRODUCTION

In this article we will consider the scattering of incident time-harmonic Rayleigh surface waves by a compact surface defect in an elastic half space. In seismological applications, and in the field of non-destructive testing when the surface imperfection is very small (e.g. in investigations with acoustic microscopes (Briggs [1]), it is often the case that the elastic waves have wavelengths much greater than a typical length scale of a defect, a say. We can therefore define a dimensionless small parameter

$$\epsilon = ka \ll 1, \tag{1}$$

say, where k is the wavenumber of compressional waves in the body. We will exploit (1) to solve the plane-strain scattering problem by the method of matched asymptotic expansions. This involves solving an 'outer problem', scaled on the wavelength $2\pi/k$, in the form of an asymptotic expansion; each term of which is forced by fundamental sources placed in the surface of the elastic material. These are compressional and shear multipoles (see Brind & Wickham [2]). Previous studies employing matched asymptotic expansions have mainly been concerned with the *scalar* problem for horizontally polarized shear waves, or, when the general elastic wave equation has been considered, attention has focussed on submerged scatterers or slowly varying surface deviations (cf. Datta & Sabina [3]).

We will confine our attention in this paper to two types of surface defect. These are the inclined planar edge crack (figure 1) and the semi-circular 'bite' (figure 2). Rescaling on the defect lenthscale, a, allows us to obtain an asymptotic expansion in the 'inner region' for either problem. Each term in these expansions is posed as a problem in elastostatics which belongs to a small class admitting explicit solution. The edge crack inner problem is solved by the Wiener-Hopf technique (Khrapkov [4]) whereas a conformal transformation allows the latter geometry to be tackled (Green & Zerna [5]). Brief details will be given for the solution in either case, and the leading order terms obtained. These terms will be shown to 'match' with the leading order outer potential and by this means we determine the coefficient of the source term of this potential. Finally we present the reflection and transmission coefficients of the scattered Rayleigh waves. Full details of the edge-crack problem, solved to the first three orders, can be found in a forthcoming paper by the authors (Abrahams & Wickham [6]).

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2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

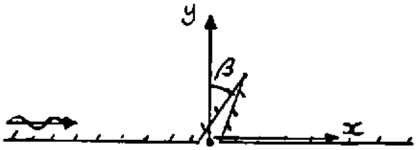

Figure 1

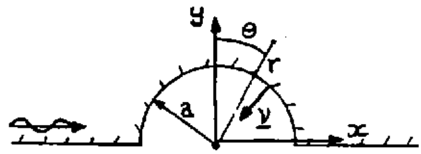

Figure 2

Defining cartesian coordinates (x, y), where the elastic body occupies the half-space $-\infty < x < \infty$, $y > 0$ excluding the defect, the elastic displacement vector $\mathbf{u}$ satisfies (after removing the harmonic time dependence factor $e^{-i\omega t}$)

$$-K^{-2}\operatorname{curl}\operatorname{curl}\mathbf{u} + k^{-2}\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{u} = 0. \tag{2}$$

Here k and K are the wavenumbers of the compressional and shear body waves respectively. On the free surface, $y = 0$, of the material the boundary conditions are

$$\sigma_{i2}(x, 0) = 0 \tag{3}$$

where $\sigma_{ij}(x, y)$ is the stress tensor and throughout we shall identify the x and y directions with the subscripts 1 and 2 respectively. On the crack or 'bite' surfaces, denoted by ∂C, we also have no tractions so

$$\sigma_{ij}\nu_j = 0, \quad \mathbf{r} \in \partial C \tag{4}$$

where ν is the outward normal vector on the void boundary and $\mathbf{r}$ is the position vector.

We now express $\mathbf{u}$ in the form

$$\mathbf{u} = \mathbf{u}^{(i)} + \mathbf{u}^{(s)}, \tag{5}$$

where the incident Rayleigh wave is written as

$$\mathbf{u}^{(i)} = U_o \begin{pmatrix} 1 & -\frac{\delta_o B_+}{ik_o A_+} \\ -\frac{\gamma_o}{ik_o} & -\frac{B_+}{A_+} \end{pmatrix} \begin{pmatrix} e^{-\gamma_o y} \\ e^{-\delta_o y} \end{pmatrix} e^{ik_o x}, \tag{6}$$

in which k_o is the wavenumber of the Rayleigh wave given by the positive real zero of

$$R(\alpha) = (2\alpha^2 - K^2)^2 - 4\alpha^2\gamma(\alpha)\delta(\alpha), \tag{7}$$

and

$$\gamma(\alpha) = (\alpha^2 - k^2)^{\frac{1}{2}} \quad , \qquad \delta(\alpha) = (\alpha^2 - K^2)^{\frac{1}{2}}, \tag{8}$$

$$\gamma_o = \gamma(k_o) \quad , \qquad \delta_o = \delta(k_o). \tag{9}$$

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To fully specify the boundary value problem we require that the scattered displacement, $\mathbf{u}^{(s)}$, satisfies a radiation condition at infinity. This can be expressed mathematically by demanding that the Lamé potentials $(\phi^{(s)}, \psi^{(s)})$, defined as

$$\mathbf{u}^{(s)} = \mathrm{grad}(\phi^{(s)}) + \mathrm{curl}(\hat{\mathbf{z}}\,\psi^{(s)}) \tag{10}$$

($\hat{\mathbf{z}}$ perpendicular to (x, y) plane), behave as

$$\phi^{(s)} = \frac{U_o}{ik_o} A_\pm e^{\pm ik_o x - \gamma_o y} + A(\theta)\frac{e^{ikr}}{(kr)^{\frac{1}{2}}} + o(kr)^{-\frac{1}{2}}, \tag{11}$$

$$\psi^{(s)} = \frac{U_o}{ik_o} B_\pm e^{\pm ik_o x - \delta_o y} + B(\theta)\frac{e^{iKr}}{(Kr)^{\frac{1}{2}}} + o(Kr)^{-\frac{1}{2}}, \tag{12}$$

as $r \to \infty$. Note that $\pm$ refer to $x > 0$ and $x < 0$ respectively, r and θ are shown in figure 2, and

$$\frac{B_\pm}{A_\pm} = \mp\frac{(2k_o^2 - K^2)}{2ik_o\delta_o}. \tag{13}$$

It is the object of this paper to find the leading order behaviour of the reflection (A_-) and transmission (A_+) coefficients of the scattered Rayleigh wave. Finally, to ensure uniqueness we insist that the displacement is bounded at the crack or bite edges.

3. OUTER POTENTIAL

Firstly we introduce the stress tensor $\Sigma_{ij}^P(\mathbf{r}, \mathbf{r}')$ corresponding to an isotropic compressional line source situated at $\mathbf{r}'$ in an elastic half space whose surface is supposed traction free. Hence on $y = 0$ we know

$$\Sigma_{i2}^P((x,0), \mathbf{r}') = 0. \tag{14}$$

By application of Green's theorem to the elastic body under investigation, we may show that

$$\mu K^2 \phi^{(s)}(\mathbf{r}') = \int_{\partial C} u_i(\mathbf{r})\Sigma_{ij}^P(\mathbf{r}, \mathbf{r}')\nu_j \, dl, \tag{15}$$

where ∂C is the boundary of either imperfection, dl is an infinitessimal increment of arc length along this boundary and μ is the shear modulus of the material. A similar expression may be obtained for the scattered shear wave potential.

We now define inner and outer coordinates according to

$$\mathbf{R} = (X, Y) = (x/a, y/a), \quad \tilde{\mathbf{r}} = (\tilde{x}, \tilde{y}) = (kx, ky) \tag{16}$$

respectively, and we further express the incident and scattered displacements in the inner region as

$$\mathbf{u}(\mathbf{r}) = U_o\left[\mathbf{U}^{(i)}(\mathbf{R}) + \epsilon\mathbf{U}(\mathbf{R})\right] \tag{17}$$

respectively. On ∂C the distance from the origin, r, is always of order a. Hence we may write $dl = adL$ and expand the stress tensor in a Taylor series expansion

$$\Sigma_{ij}^{P}(a\mathbf{R},\bar{\mathbf{r}}'/k) = \Sigma_{ij}^{P}(0,\bar{\mathbf{r}}'/k) + \epsilon X_k \frac{\partial}{\partial \bar{x}_k}\Sigma_{ij}^{P}(\mathbf{r},\bar{\mathbf{r}}'/k) + \dots \quad (18)$$

provided that $\bar{r}' > \epsilon$. Therefore, the outer solution behaves as

$$\begin{aligned}\phi^{(s)}(\bar{\mathbf{r}}'/k) &= \frac{\epsilon^2 U_o}{\mu K^2 k}\left\{\Sigma_{11}^{P}(0,\bar{\mathbf{r}}'/k)\int_{\partial C} U_1(\mathbf{R})\nu_1\, dL\right. \\ &+ \left.\Sigma_{11}^{P}(0,\bar{\mathbf{r}}'/k)\int_{\partial C} U_1^{(i)}(\mathbf{R})\nu_1\, dL + \frac{\partial}{\partial \bar{x}_k}\Sigma_{ij}^{P}(0,\bar{\mathbf{r}}'/k)\int_{\partial C} U_i^{(i)}(\mathbf{R})\nu_j X_k\, dL\right\} + O(\epsilon^3),\end{aligned} \quad (19)$$

where, from Brind & Wickham [2],

$$\frac{1}{\mu K^2}\Sigma_{11}^{P}(0,\mathbf{r}) = -\frac{4(\tau^2-1)}{\pi\tau^2}\int_{-\infty}^{\infty}\frac{s^2\delta(s)}{R(s)}e^{-isx-\gamma(s)y}\, ds \quad (20)$$

etc., and $\tau = K/k$. The first integral term in (19) is as yet unknown, and it is this constant which must be determined from an examination of the inner region. The latter terms are integrals involving the forcing, and are therefore known from (6). For the edge crack we can easily show that the forcing integrals are identically zero whereas the first integral may be written as

$$D_e = \cos\beta\int_0^1 U_1(R,\beta)|_-^+\, dR, \quad (21)$$

where $|_-^+$ denotes the jump across the crack from the left to right face as shown (figure 1). Similarly for the semi-circular void we know

$$\begin{aligned}\phi^{(s)}(\bar{\mathbf{r}}'/k) &\sim \frac{\epsilon^2 U_o}{\mu K^2 k}\left\{\Sigma_{11}^{P}(0,\bar{\mathbf{r}}'/k)D_b - \frac{\pi}{2}\left[U_2^{(i)}(0)\frac{\partial}{\partial \bar{x}_2}\Sigma_{22}^{P}(0,\bar{\mathbf{r}}'/k)\right.\right. \\ &+ \left.\left. U_1^{(i)}(0)\frac{\partial}{\partial \bar{x}_1}\Sigma_{11}^{P}(0,\bar{\mathbf{r}}'/k) + U_1^{(i)}(0)\frac{\partial}{\partial \bar{x}_2}\Sigma_{12}^{P}(0,\bar{\mathbf{r}}'/k)\right]\right\},\end{aligned} \quad (22)$$

where

$$D_b = -\int_{-\pi/2}^{\pi/2}\sin\theta\, U_1(R,\theta)\, d\theta - \frac{i\pi K^2}{4kk_o}. \quad (23)$$

By solving the respective inner problems to find U_1 we could determine the constants from the integral identities (21), (23). It is simpler however to match inner and outer solutions together as $R \to \infty$ and $\bar{r} \to 0$. We may obtain the form of the outer expansion as $\bar{r} \to 0$ from (20) and by referring to [6]. We find

$$\phi^{(s)}(\bar{r}/k) \sim \frac{4\epsilon^2 U_o D}{\pi k\tau^2}\frac{\cos 2\theta}{\bar{r}^2}. \quad (24)$$

Note that the presence of the free surface strengthens the singularity. If the defect was some distance below the surface then one would obtain $\log\bar{r}$ behaviour close to it.

4. INNER EDGE CRACK PROBLEM

The governing equation in the inner region is obtained by rescaling the dependent and independent variables. To leading order we find

$$-K^2 \mathrm{curl}_X\,\mathrm{curl}_X \mathbf{u} + k^2 \mathrm{grad}_X\,\mathrm{div}_X \mathbf{u} = 0 \tag{25}$$

which is just the elastostatic equation. Here

$$\mathrm{div}_X \mathbf{U} = \frac{\partial U_1}{\partial X} + \frac{\partial U_2}{\partial Y} \tag{26}$$

etc. On the crack we know that the stresses, written in inner coordinates, are

$$\sigma_{R\theta} = -2i\mu(k/k_o)(\tau^2-1)\sin\beta\cos\beta, \quad \theta=\beta,\ R<1, \tag{27}$$

$$\sigma_{\theta\theta} = -2i\mu(k/k_o)(\tau^2-1)\cos^2\beta, \quad \theta=\beta,\ R<1, \tag{28}$$

and the normal and shear stresses are zero on $Y = O$. We also insist that the displacements are bounded everywhere.

The boundary value problem may be formulated as a matrix Wiener-Hopf equation which can be solved exactly. The details may be found in Abrahams & Wickham [6]; here we simply quote the solution for the sum of the principal stresses as $R \to \infty$:

$$\Theta = \sigma_{XX} + \sigma_{YY} \sim \frac{8i\mu k(\tau^2-1)}{k_o\pi}\cos^2\beta\, C_1(\beta)\frac{\cos 2\theta}{R^2}, \tag{29}$$

$$C_1(\beta) = (\cos\beta, \sin\beta)\mathbf{Q}_+(2)\mathbf{Q}_-(0)\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}, \tag{30}$$

where $\mathbf{Q}_+(\alpha)$, $\mathbf{Q}_-(\alpha)$ are the matrix multiplicative factors of

$$\frac{\sin\alpha\pi}{\Delta(\alpha)}(n(\alpha)\mathbf{I} + m(\alpha)\mathbf{J}(\alpha)). \tag{31}$$

In (31) I is the identity,

$$n(\alpha) = \cos[2(\alpha-1)\beta] + \cos(\alpha\pi) - 2(\alpha-1)^2\cos^2\beta\cos[2(\alpha-1)\beta], \tag{32}$$

$$m(\alpha) = 2(\alpha-1)\cos\beta\sin[2(\alpha-1)\beta], \tag{33}$$

$$\Delta(\alpha) = n^2(\alpha) - m^2(\alpha)(1-(\alpha-1)^2\cos^2\beta) \tag{34}$$

and

$$\mathbf{J}(\alpha) = \begin{pmatrix}\sin(\beta) & (\alpha-2)\cos\beta\\ -\alpha\cos(\beta) & -\sin\beta\end{pmatrix}. \tag{35}$$

Here $Q_+(\alpha)$ is non-zero and has no singularities in $\Re(\alpha) > 1$, and similarly $Q_-(\alpha)$ is non-zero and has no singularities in $\Re(\alpha) < 2$. Both factors behave like constant matrices at infinity in their respective half planes of analyticity.

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5. INNER SEMI-CIRCULAR VOID PROBLEM

To leading order, the inner problem has the governing equation written in (25) but now the boundary conditions on the curved surface are

$$\sigma_{R\theta} = -2i\mu(k/k_o)(\tau^2-1)\sin\theta\cos\theta, \quad -\pi/2 \le \theta \le \pi/2,\ R=1, \tag{36}$$
$$\sigma_{\theta\theta} = -2i\mu(k/k_o)(\tau^2-1)\sin^2\beta, \quad -\pi/2 \le \theta \le \pi/2,\ R=1. \tag{37}$$

This problem is most conveniently solved by working with the Airy stress function $\chi(\mathbf{R})$ and employing a conformal transformation. We use the mapping

$$z = X + iY = \coth(\frac{1}{2}\zeta) = \coth\frac{1}{2}(\xi + i\eta) \tag{38}$$

which transforms any point on the straight boundary $Y=0$ to a point on the line $\eta = 0$, whilst the arc $R=1$ maps to $\eta = -\pi/2$ (Green & Zerna [5]). Hence the elastic body is mapped into an infinite strip, a geometry which can then be tackled by Fourier transforms. Again omitting the details we find that

$$\chi(\zeta) = -\frac{i\mu(k/k_o)(\tau^2-1)}{2(\cosh\xi - \cos\eta)}\int_{-\infty}^{\infty}\frac{T(s)e^{-is\xi}}{(\sinh^2(s\pi/2)-s^2)}ds, \tag{39}$$

where

$$T(s) = \sin\eta\sinh s\eta[(1+s^2)\tanh(s\pi/2) + s^2\coth(s\pi/2)] - s[\cos\eta\sinh s\eta - s\sin\eta\cosh s\eta]. \tag{40}$$

We can determine the trace of the stress tensor as $R \to \infty$ from this result, which for mathematical convenience we take along the line $Y=0$ only. Thus

$$\Theta(X,0) \sim -\frac{4i\mu(k/k_o)(\tau^2-1)}{X^2}C_2, \quad X \to \pm\infty, \tag{41}$$

where

$$C_2 = 1 + 2\int_0^{\infty} s^3\frac{\coth(s\pi/2)}{(\sinh^2(s\pi/2)-s^2)}ds \approx 2.0850. \tag{42}$$

6. MATCHING AND CONCLUDING REMARKS

In order to determine the two unknown ceofficients D_e, D_b from section 3 we must match the outer solution as $\bar{r} \to 0$, (24), with the inner solutions, (29), (41), as $R \to \infty$. To achieve this we must establish a relationship between the outer compressional potential $\phi^{(s)}$ and the inner stress invariant Θ. We can show that

$$\Theta = -\frac{2\mu k}{U_o}(\tau^2-1)\phi^{(s)}, \tag{43}$$

and so for the edge crack we find that (replacing $\bar{r}$ by ϵR in (24))

$$D_c = -i(k/k_o)\tau^2\cos^2\beta\, C_1(\beta). \tag{44}$$

For the 'bite' we match along $\theta = \pm\pi/2$ which gives

$$D_b = -\frac{i\pi K^2}{2kk_o}C_2. \tag{45}$$

This completes the leading order solutions and from these values of D we can determine the scattered Rayleigh wave coefficients. For the semicylindrical groove we find

$$A_\pm = \frac{8k_o^3\delta_o(\tau^2-1)D_b}{k\tau^2R'(k_o)}\epsilon^2 + \frac{\pi i\tau^2K^2}{8k_o^2\delta_oR'(k_o)}\left\{(2k_o^2-K^2)^2(1+\mathrm{sgn}(x)) - K^4\mathrm{sgn}(x)\right\}\epsilon^2, \tag{46}$$

where R' is the derivative of the Rayleigh determinant written in (7), and for the edge-crack

$$A_\pm = \frac{8k_o^3\delta_o(\tau^2-1)D_c}{k\tau^2R'(k_o)}\epsilon^2. \tag{47}$$

There are several points to note. Firstly the scattered surface waves are very small, $O(\epsilon^2)$, which is not too surprising in view of the fact that the imperfection is small. Secondly we see that, to leading order, the reflection and transmission coefficients are identical in the case of the edge crack, although clearly the geometry is not symmetrical. The angle at which the crack is inclined to the vertical affects the magnitude of the coefficients through $C_1(\beta)$, but the asymmetry in the scattered waves will only appear at next order in the expansions. This is in contrast to the 'bite' geometry, which gives rise to asymmetrical scattering at leading order. The authors are currently comparing the wave coefficients calculated for the latter problem with results given recently by Gregory and Austin [7].

Another point to raise regarding the semicircular void is that the coefficients $A_\pm$ are purely imaginary. The could lead to the *obviously false* impression, when performing an energy balance, that the Rayleigh wave propagating to $+\infty$ has more energy than the incident wave! This possible misunderstanding can be cleared by performing matching to next order $O(\epsilon^3)$ in which A_+ has a real component. This will then show that the total energy of the right travelling Rayleigh wave (in $x > 0$) is smaller than that of the forcing.

The method presented in this paper can easily be employed for a wide range of surface breaking or near surface imperfections. For traction free defects the outer expansion will be identical to those given in section 3. For inner problems in which the displacements are specified an alternative form of the potential to that in (15) may be deduced, and, this will lead to a different form for the outer expansion.

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7. REFERENCES

[1] G. A. D. Briggs, 'How fine a surface crack can you see in a scanning acoustic microscope?', in *IUTAM Symposium on Elastic Waves and Ultrasonic Nondestructive Evaluation*, eds. S. K. Datta and J. D. Achenbach, North-Holland (1990)

[2] R. J. Brind & G. R. Wickham, 'Near-field behaviour of the fundamental elastodynamic solutions for a semi-infinite homogeneous isotropic elastic solid', *Proc Roy Soc Lond A*, to be published (1991)

[3] S. K. Datta & F. Sabina, 'Matched asymptotic expansions applied to diffraction and high frequency asymptotics', in *High and Low frequency asymptotics*, eds. V. K. Varadan & V. V. Varadan, North-Holland (1986)

[4] A. A. Khrapkov, 'Certain cases of elastic equilibrium of an infinite wedge with a nonsymmetric notch at the vertex, subjected to concentrated forces', *Prikl Mat Mekh*, **35** pp677-689 (1971)

[5] A. E. Green & W. Zerna, *Theoretical elasticity*, Clarendon Press (1954)

[6] I. D. Abrahams and G. R. Wickham, 'Scattering of surface or body waves by a small inclusion in the surface of an elastodynamic half-space', *J Mech Phys Solids*, to be submitted (1991)

[7] R. D. Gregory and D. M. Austin, 'Scattering of waves by a semicylindrical groove in the surface of an elastic half-space', *Q J Mech Appl Math*, **43**(3) pp293-315 (1990)