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INTERACTION OF PERFORATIONS IN AN ACOUSTIC WAVEGUIDE AT LOW FREQUENCIES

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1. INTRODUCTION

The problem of interacting discontinuities in a waveguide occurs when the distance between two discontinuities is smaller than the transverse dimensions of the guide, the coupling being due to evanescent modes. For perforations between two waveguides (and the application to perforated tube mufflers), the theoretical problem is not simple because the complete model of a perforation includes both symmetrical mode effects (a shunt inductance) and antisymmetrical mode effects (see ref.1). A similar problem is the problem of interacting branched tubes into a waveguide (see figure 1), an application being the side holes on woodwind musical instruments. We will here discuss the latter problem and show that the general result is effectively rather complicated, but an approximate result can show the effect of interaction on the equivalent circuit without interaction in a rather simple way. The main result is the theoretical treatment of the general case where overlap between holes can occur (see fig.1). Relation with radiation is discussed in the conclusion.
The method is based on an integral equation and modal decomposition, written using a matricial formalism (see refs. 2 and 3), equivalent to the integral formalism (see ref.4).

2. THE FIELD IN THE MAIN GUIDE : THE MATRICIAL NOTATION

In the main guide, the pressure field is decomposed into the incident and reflected fields, as follows :

$$p(r) = {}^t\psi(w)\ [E^+(z)P^+ + E^-(z)P^-]$$

where $\psi(w)$ is the (column) vector of the eigenmodes of the wave equation in the guide ; P^+ and P^- are the (column) vectors of the complex amplitudes of the modes, and $E^{\pm}(z)$ are diagonal matrices defined by : $E^{\pm}_{ii}(z) = e^{\mp jk_i z}$
where $k_i = (k^2 - \gamma_i^2)^{1/2}$ is the wavenumber of mode i, $k = \omega/c$, γ_i is the eigenvalue of mode i.

The coefficients P^+ and P^- differ in the three parts of the main guide : on the left of hole A, between holes A and B, on the right of hole B. If the holes are sufficiently far from other discontinuities, P^+_{LA} and P^-_{RB} are reduced to the propagating modes (the planar mode only at low frequencies). It is convenient to define the coefficients with respect to the centers of the holes (z_A and z_B), i.e. :

$$p(r) = {}^t\psi(w)\ [E^+(z-z_A)P^+_{LA} + E^-(z-z_A)\ P^-_{LA}] \qquad \text{to the left of hole A} \quad (1)$$

$$p(r) = {}^t\psi(w)\ [E^+(z-z_B)P^+_{RB} + E^-(z-z_B)\ P^-_{RB}] \qquad \text{to the right of hole B} \quad (2)$$

The diagonal matrix of the characteristic impedances is defined as follows :

$$Z_{c_i} = \frac{\rho c}{S}\ \frac{k}{k_i}\ . \tag{3}$$

3. THE INTEGRAL EQUATION

If one considers the volume V to be bounded by the waveguide walls and the areas of the branched tubes, S_A and S_B (see fig.1), one can write the Helmholtz-Huyghens integral equation for the volume V.

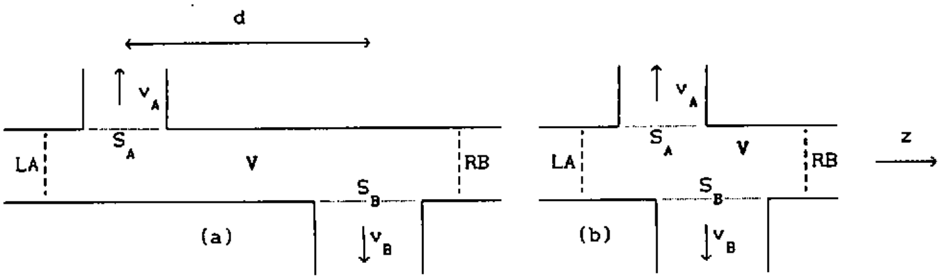

figure 1 : geometry of the waveguide

(a) case without overlap, (b) case with overlap.

By projecting the integral equation on the surfaces LA, RB, S_A, S_B and decomposing the field on the modes ψ_A and ψ_B of the surfaces S_A and S_B, respectively, one obtains the following equations :

$$p^-_{LA} = -\frac{1}{2}\ z_c\ {}^tq^+_A\ U_A - \frac{1}{2}\ z_c e^-(z_A-z_B)\ {}^tq^+_B\ U_B + e^-(z_A-z_B)p^-_{RB} \tag{4a}$$

$$p^+_{RB} = -\frac{1}{2}\ z_c\ e^+(z_B-z_A)\ {}^tq^-_A\ U_A - \frac{1}{2}\ z_c\ {}^tq^-_B\ U_B + e^+(z_B-z_A)p^+_{LA} \tag{4b}$$

$$P_A = -Z_{AA}U_A - Z_{AB}U_B + q^+_A p^+_{LA} + q^-_A e^-(z_A-z_B)p^-_{RB} \tag{4c}$$

$$P_B = -Z_{BA}U_A - Z_{BB}U_B + q^+_B e^+(z_B-z_A)p^+_{LA} + q^-_B p^-_{RB} \tag{4d}$$

where p^-_{LA}, p^+_{RB}, p^-_{RB}, p^+_{LA} are the planar mode coefficients of the vectors defined in equations (1) and (2) (p^-_{RB} and p^+_{LA} are the source quantities), P_A, P_B, U_A, U_B are the pressure and outward velocity vectors for the surfaces S_A and S_B (for U_A and U_B the velocities are multiplied by their respective area). The other quantities are :

- $z_c = \rho c/S$, $e^{\pm}(z) = e^{\mp jkz}$,
- $q^{\pm}_A = \frac{1}{S_A}\int_{S_A} e^{\pm}(z-z_A)\psi_A dS$, $q^{\pm}_B = \frac{1}{S_B}\int_{S_B} e^{\pm}(z-z_B)\psi_B dS$,
- $Z_{ij} = \frac{j\omega\rho}{S_i S_j}\int_{S_i}\int_{S_j} \psi_i(r)G(r,r')^t\psi_j(r')dS_i dS'_j$ (i, j = A or B),
- $G(r,r')$ is the Green function for the infinite guide.

4. THE SEPARATION OF THE TWO HOLES

The equations (4) do not allow to exhibit the limit of no coupling by evanescent modes occurs. This result is obtained directly by using the description for each hole separately (see refs.1,3), and the junction between the two holes through the planar mode in the main guide. The equivalent circuit is shown in fig.2.

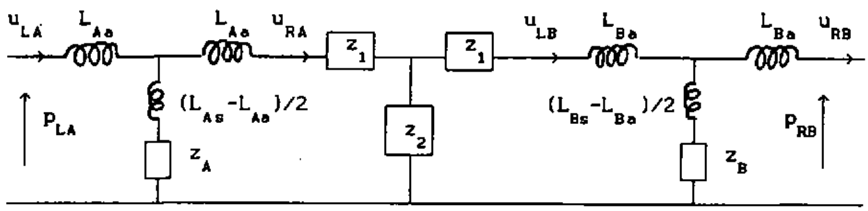

figure 2 : equivalent circuit for the two holes when no coupling occurs

$$z_1 = z_c jtg(\frac{kd}{2}),\quad z_2 = z_c^{-1} jsin(kd).$$

Thus it is convenient to use intermediate quantities p^-_{RA}, p^+_{RA}, p^-_{LB}, p^+_{LB} defined as follows :

$$p^-_{LA} = -\frac{1}{2} z_c {}^t q^+_A U_A + p^-_{RA} \quad (5a) \qquad p^+_{RA} = -\frac{1}{2} z_c {}^t q^-_A U_A + p^+_{LA} \quad (6a)$$

$$p^-_{LB} = -\frac{1}{2} z_c {}^t q^+_B U_B + p^-_{RB} \quad (5b) \qquad p^+_{RB} = -\frac{1}{2} z_c {}^t q^-_B U_B + p^+_{LB} \quad (6b)$$

In the case where no overlap occurs between the two holes, these quantities correspond to the coefficients for the planar mode to the right of hole A and to the left of hole B. The important feature is that these quantities can also be defined for the more general case where overlap occurs, and equations (5a), (5b), (6a), (6b) are equivalent to equations (4a) and (4b) if :

$$p^{\pm}_{RA} = e^{\pm}(z_A - z_B)p^{\pm}_{LB}. \qquad (7)$$

Then, by using equations (4c), (5b), (7), one obtains :

$$P_A = - Z_{AA}U_A - H_{AB}U_B + q^+_A p^+_{LA} + q^-_A p^-_{RA} \qquad (8a)$$

and, similary :

$$P_B = - H_{BA}U_A - Z_{BB}U_B + q^+_B p^+_{LB} + q^-_B p^-_{RB}, \qquad (8a)$$

where $H_{AB} = Z_{AB} - \frac{1}{2} z_c q^-_A e^-(z_A - z_B)\,{}^t q^+_B$ and $H_{BA} = {}^t H_{AB}$.

By decomposing the Green function G into the planar mode and the higher modes (G'), H_{AB} can be rearranged as follows :

$$H_{AB} = \frac{j\omega\rho}{S_A S_B}\int_{S_A}\int_{S_B} \psi_A(r)G'(r,r')\psi_B(r')dS_A dS'_B \qquad (9)$$

$$- \frac{jz_c}{2S_A S_B}\int_{S_A}\int_{S_B} \psi\,(r)\Big[\sin(k|z-z'|) + \sin[k(z-z')]\Big]{}^t\psi_B(r')dS_A dS_B'.$$

Some interesting properties of this result will be discussed in section 5. It remains to rearrange equations (5a), (5b), (6a), (6b), (8a), (8b) in order to exhibit pressure and volume velocity of the plane mode (p_{LA}, u_{LA}, p_{RA}, u_{RA}, p_{LB}, u_{LB}, p_{RB}, u_{RB}). It is made in the same way as for the problem of a single hole (see refs 1,3). Finally, one obtains :

$$\left\{\begin{array}{ll} p_{LA} - p_{RA} = jz_c\,{}^t\beta_A U_A & p_{LB} - p_{RB} = jz_c\,{}^t\beta_B U_B \\ u_{LA} - u_{RA} = {}^t\alpha_A U_A & u_{LB} - u_{RB} = {}^t\alpha_B U_B \\ \multicolumn{2}{l}{P_A = - H_{AA}U_A - H_{AB}U_B + \frac{1}{2}\alpha_A(p_{LA} + p_{RA}) - \frac{1}{2} jz_c\beta_A(u_{LA} + u_{RA})} \\ \multicolumn{2}{l}{P_B = - H_{BA}U_A - H_{BB}U_B + \frac{1}{2}\alpha_A(p_{LB} + p_{RB}) - \frac{1}{2} jz_c\beta_B(u_{LB} + u_{RB})} \end{array}\right. \qquad (10)$$

where • H_{AA} and H_{AB} are obtained from Z_{AA} and Z_{BB} by replacing G by Re(G),

• α and β are defined by : $2\alpha = q^+ + q^-$, $2j\beta = q^+ - q^-$.

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5. DISCUSSION

The known results for a single hole can be obtained by setting either $U_B=0$ or $U_A=0$, respectively. For two holes without overlap, the second integral of H_{AB} in equation (9) vanishes, and coupling occurs only through the evanescent modes of the main guide : if the distance (z_A-z_B) tends to infinity, G' tend to zero and H_{AB} too.

We know that all matrices H_{ij} are inductive, i.e. proportional to $j\omega$ for low frequencies. If the holes have a plane of symmetry, thus the modes ψ_A and ψ_B can be divided into symmetrical and antisymmetrical modes, the vectors α and β being zero for the antisymmetrical and symmetrical modes, respectively.

There is no interaction between symmetrical and antisymmetrical modes for a given hole, i.e. H_{AA} and H_{BB} can be divided into four submatrices, the two submatrices of the second diagonal being zero. On the contrary, H_{AB} does not involve zeros.

6. SOLUTIONS OF THE SET OF EQUATIONS

If the holes are terminated in known impedances, the input impedance matrices (or the Green function) are known :

$$\mathbf{P}_A = \mathbf{Z}_A\mathbf{U}_A \qquad \text{and} \qquad \mathbf{P}_B = \mathbf{Z}_B\mathbf{U}_B. \tag{11}$$

These equations, with equations (10) and (7), allow one to calculate the transfer matrix for the planar mode relating the quantities (p_{LA}, u_{LA}) to (p_{RB}, u_{RB}). This is in general a tedious calculation. Our aim is to give an answer to the following question : is it possible to simply represent the effect of the interaction terms (H_{AB}) on the equivalent circuit obtained when no interaction occurs, i.e. when the interhole distance d is large (see fig.2) ? We think that the answer is no in general.

Nevertheless, we can restrict our study to the case of open holes (thus Z_A and Z_B are inductive, the planar mode impedance z_A and z_B including the radiation impedance, see ref.4) and low frequencies : thus the planar mode equations (see equation 7) in the main guide are :

$$\begin{pmatrix} p_{RA} \\ u_{RA} \end{pmatrix} = \begin{pmatrix} 1 & j\omega\rho d/S \\ j\omega dS/\rho c^2 & 1 \end{pmatrix} \begin{pmatrix} p_{LB} \\ u_{LB} \end{pmatrix}.$$

If we introduce these equations into equations (10), it is easy to show that the capacitive term ($j\omega dS/\rho c^2$) can be ignored, because it is always in parallel with inductive terms. Thus the planar mode equations are simply :

$$p_{RA} - p_{LB} = \frac{j\omega\rho d}{S} u \qquad\qquad u_{RA} = u_{LB} = u$$

Now, we see that all the impedances in equations (10) are inductive, thus the form of the final result is :

$$\begin{pmatrix} p_{LA} \\ p_{RB} \end{pmatrix} = \begin{pmatrix} j\omega L_{AA} & -j\omega L_{AB} \\ j\omega L_{BA} & -j\omega L_{BB} \end{pmatrix} \begin{pmatrix} u_{LA} \\ u_{RB} \end{pmatrix} \qquad (12)$$

and the equivalent circuit as shown in fig.3.

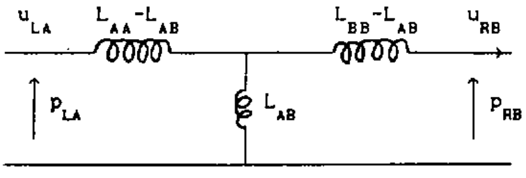

figure 3 : equivalent circuit for the two holes when coupling occurs.

Nevertheless, the expressions of the quantities L_{ij} are very complicated ; therefore, we are searching approximations. As a matter of fact :

i) for one hole, the effect of antisymmetrical modes is very small compared to the effect of symmetrical modes (see ref.5).

ii) for two holes, the effect of interaction (H_{AB}) can be regarded as a perturbation of the result for large interhole distance d.

Thus, we can consider that the effect of antisymmetrical modes for one hole (L_{AA} and L_{BA} in fig.2) and the effect of interaction are small perturbations of the simplest, classical result for open side holes. We notice that these approximations depend only on the geometry and are independant of frequency. Thus, we can show that the final result is the equivalent circuit shown in fig.4, the effect of the interaction being the inductance L_{int}, given by the following expression :

$$1/\ j\omega L_{int} = {}^{t}\alpha_A (Z_A + H_{AA})^{-1}\ H_{AB} (Z_B + H_{BB})^{-1} \alpha_B\ .$$

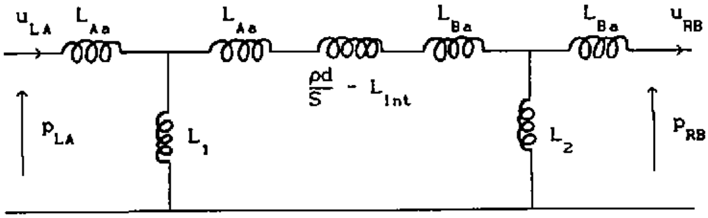

figure 4 : equivalent circuit for the two holes when coupling occurs,
$L_1 = \frac{1}{2}(L_{As}-L_{Aa}) + L_{Ah} + L_{int}$, $L_2 = \frac{1}{2}(L_{Bs}-L_{Ba}) + L_{Bh} + L_{int}$.

7. CONCLUSION

As a consequence, we see that the effect of interaction is mainly the modification of the "inner length correction" (inductance L_{As}), and of the planar mode inductance $\rho d/S$. Nevertheless, at low frequencies, the effect of shunt inductances is more important than the effect of series inductances, thus the main effect of interaction is the modification of the inductances L_{As}, which can be regarded as a radiation impedance of a hole into the main guide. This result is not a surprise (it remains to verify if this modification is negative, as for radiation impedances, or positive). But our derivation shows that the complete effect is rather complicated, and is reduced to analogous effects for radiation impedances only as an approximation. As a matter of fact, the present study can be directly applied to the so-called "flute" mode of the lattice constituted by two waveguides coupled by perforations (the other mode being a planar mode, see ref.1). Moreover, concerning the open side holes of woodwind instruments, the present problem is the internal interaction, but another problem occurs as well, that of the external interaction (through radiation into infinite space). For this latter problem, it can be shown that, if the antisymmetrical modes effect is ignored, the equivalent circuit has the shape of the circuit of figure 3, but resistances need to be added in parallel with each inductance.

8. REFERENCES

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