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ANALYTICAL SOLUTIONS OF THE ONE-DIMENSIONAL ACOUSTIC WAVES IN A DUCT

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Wave-based models for one-dimensional duct acoustics are widely used in thermoacoustic network models. However, they currently assume a constant mean temperature and mean flow within each duct module, while in practice many ducts of relevance sustain a significant axial temperature gradient or mean flow gradient. This paper presents an analytical solution for the one-dimensional acoustic field in a duct with arbitrary mean temperature gradient and mean flow. A wave equation for the pressure perturbation is derived which relies on very few assumptions. An analytical solution for this is derived using an adapted WKB approximation. The proposed solution is applied to ducts with a mean temperature profile which varies axially with (i) a linear and (ii) a partial sine wave profile. The analytical solution reproduces the acoustic field very accurately across a wide range of flow conditions which span both low and moderate-to-high subsonic Mach numbers. It always performs well when the frequency exceeds a certain value; when the mean temperature profile is linear, it also performs well to very low frequencies. This increased frequency range for linear mean temperature profiles leads to its application to more complicated profiles in a piecewise linear manner, axially segmenting the temperature profile into regions that can be approximated as linear. The acoustic field is predicted very accurately as long as enough segmentation points are used and the condition for the linear mean temperature profile is satisfied: $|k_0| > |\alpha|$, where k_0 is the local wave number when there is no mean flow and α is the normalised mean density gradient.

Keywords: Arbitrary mean axial temperature gradient; Mean flow; WKB approximation; Piecewise linear approximation

1. Introduction

It would be valuable to have an accurate analytical solution for the one-dimensional acoustic field within ducts sustaining both a mean flow and a mean axial temperature gradient, particularly if the solution could be represented as the superposition of waves travelling in either direction. This would offer enhanced physical insight, reduce the computational cost of numerical tools used to predict[1, 2, 3] and control[4] thermoacoustic instabilities, and would bring benefits for measurement approaches based on the two-microphone [5], and reproduction of underwater thermodynamic properties based on the solution of the inverse scattering problem in one dimension [6].

There are generally two kinds of approaches for deriving the analytical solution of the one-dimensional acoustic field. The first is based on variable transformation. The wave equation is transformed to a standard second order ordinary differential equations with known solutions [7]. The second category uses linear perturbation theory and assumes that the acoustic wave equation consists of wave-like solutions for slowly varying coefficients of the ordinary differential equation [8]. However, the above solutions are all limited to small linear mean temperature gradients and low mean flow Mach numbers. Cummmings [9] derived an analytical solution using an adapted WKB approximation and assuming sufficiently large frequencies and the absence of mean flow. This WKB approximation method was extended to account for mean flow [10]. However, too many terms were omitted in the wave equation and solutions are not accurate at larger mean flow Mach numbers. To summarise, no analytical or semi-analytical solution for the one-dimensional acoustic field in a duct has previously been presented which allows an arbitrary mean axial temperature gradient and a mean flow of moderate subsonic Mach number.

This work derives an acoustic wave equation which relies on very few assumptions. It then uses an adapted WKB approach to derive the analytical solution for the one-dimensional acoustic field. The proposed analytical solution is simple and applies to large mean temperature gradients and moderate-to-large mean flow Mach numbers. Furthermore, for linear mean temperature profiles, it is seen to yield a particularly simple expression and to be accurate at both low and high frequencies. This suggests that accurate prediction for arbitrary mean temperature profiles can be achieved by applying it in a piecewise linear manner to an appropriately axially segmented mean temperature profile.

The remainder of the paper is organised as follows. The derivations of the acoustic wave equation and analytical solutions are presented in Section 2 and Section 3 respectively. Section 4 introduces the linear and sine wave mean temperature profiles and the two transfer functions used for validation of the proposed analytical solution. Validation of predictions for the two mean temperature profiles are presented in Section 5. Conclusions are drawn in the final section.

2. Acoustic wave equation

One considers a constant cross-sectional area duct sustaining a mean flow and mean temperature gradient. The flow within the duct is considered perfect, inviscid and isentropic, and comprises a steady uniform time-averaged flow (denoted $\overline{()}$) and small perturbations (denoted $()'$). Assuming that all fluctuating quantities have a time dependence in the form $y' = \hat{y}e^{\mathrm{i}\omega t}$, where ω is the complex angular frequency, the linearised one-dimensional momentum and energy conservation equations along the duct give:

$$\left(\mathrm{i}\omega + \frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right)\hat{u} + \bar{u}\frac{\mathrm{d}\hat{u}}{\mathrm{d}x} + \bar{u}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\frac{\hat{p}}{\gamma\bar{p}} + \frac{1}{\bar{\rho}}\frac{\mathrm{d}\hat{p}}{\mathrm{d}x} = 0 \tag{1}$$

$$\left(\mathrm{i}\omega + \gamma\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right)\hat{p} + \bar{u}\frac{\mathrm{d}\hat{p}}{\mathrm{d}x} + \frac{\mathrm{d}\bar{p}}{\mathrm{d}x}\hat{u} + \gamma\bar{p}\frac{\mathrm{d}\hat{u}}{\mathrm{d}x} = 0 \tag{2}$$

where ρ, u, p and T denote density, axial velocity, pressure and temperature respectively. x is axial location. γ denotes the ratio of specific heats and is assumed constant along the duct. If Mach number terms of order higher than M^2 are neglected, the two equations can be combined to an acoustic wave equation which is a function only of $\hat{p}$ [11]:

$$\begin{aligned}&\left(1 - M^2 + \mathrm{i}\frac{2M^2}{k_0}\frac{\mathrm{d}M}{\mathrm{d}x}\right)\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}x^2} - \left(\left(1-(3+\gamma)M^2\right)\alpha + \mathrm{i}2Mk_0 + \mathrm{i}\frac{M\beta}{k_0} - \mathrm{i}\frac{2M\alpha^2}{k_0}\right)\frac{\mathrm{d}\hat{p}}{\mathrm{d}x}\\&+\left(k_0^2 + \mathrm{i}(2+\gamma)Mk_0\alpha - \mathrm{i}2\gamma k_0 M^2\frac{\mathrm{d}M}{\mathrm{d}x} + (2-\gamma)M^2\beta + (4\gamma-5)M^2\alpha^2\right)\hat{p} = 0\end{aligned} \tag{3}$$

where $\alpha = \frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x}$ and $\beta = \frac{1}{\bar{\rho}}\frac{\mathrm{d}^2\bar{\rho}}{\mathrm{d}x^2}$ are the normalised first and second order differential of mean density respectively. $k_0 = \omega/\bar{c}$ is the local wave number in the absence of mean flow, $\bar{c} = (\gamma R_g\overline{T})^{1/2}$ is the local speed of sound and R_g is the universal gas constant that is also considered constant.

3. Analytical solutions of the acoustic field

The acoustic wave equation defined in Eq. (3) is a second order ordinary differential equation with three variable-dependent coefficients. The WKB method provides a good approximation of the wave-like solutions of this kind of ordinary differential equation. A wave-like solution, generally consisting of an asymptotic expansion, is assumed and substituted into the differential equation. By balancing terms of different orders, the coefficients can be found, along with the approximate solution of the differential equation [12]. The current work proposes an approximation method which is based on the WKB method, but with slight changes. The assumed wave-like solution is an exponential, as in previous work. However, instead of the asymptotic expansion assumption made in the WKB method, the solution is assumed to have separate amplitude and phase factors, with the expression $\hat{p} = \mathscr{C} \exp\left(\int_{x_1}^{x} a + \mathrm{i}b \, \mathrm{d}x\right)$, where $\mathscr{C}$ is an arbitrary constant. a and b are two real x-dependent variables. Substituting this expression into the acoustic wave equation (Eq. (3)) leads to:

$$\left(1 - M^2 + \mathrm{i}\frac{2M^2}{k_0}\frac{\mathrm{d}M}{\mathrm{d}x}\right)\left(a^2 - b^2 + \mathrm{i}2ab + \frac{\mathrm{d}a}{\mathrm{d}x} + \mathrm{i}\frac{\mathrm{d}b}{\mathrm{d}x}\right) - \big((1-(3+\gamma)M^2)\alpha + \mathrm{i}2Mk_0 + \mathrm{i}\frac{M\beta}{k_0} - \mathrm{i}\frac{2M\alpha^2}{k_0}\big)\big(a + \mathrm{i}b\big) + k_0^2 + \mathrm{i}(2+\gamma)Mk_0\alpha - \mathrm{i}2\gamma k_0 M^2\frac{\mathrm{d}M}{\mathrm{d}x} + (2-\gamma)M^2\beta + (4\gamma - 5)M^2\alpha^2 = 0 \tag{4}$$

Considering the real part of above equation with k_0 taken as a real number (the imaginary part of k_0 is generally much smaller than the real part in a thermoacoustic system [2]), neglecting further small terms and assuming that $|k_0| \gg |\alpha|$ and $|k_0^2| \gg |\beta|/2$ yield:

$$k_0^2 - (1-M^2)b^2 + 2k_0 M b \approx 0. \tag{5}$$

Solutions for b follow by solving the above equation:

$$b^+ = -k_0/(1+M) \quad \text{and} \quad b^- = k_0/(1-M) \tag{6}$$

Considering the imaginary part of Eq. (4), neglecting very small terms and substituting the two solutions of b into it lead to the corresponding solutions of a.

- When $b = -k_0/(1+M)$, $|k_0^2| \gg M\alpha^2$ and $|k_0^2| \gg M|\beta|/2$, the simplified solution for a is:

$$a^+ = \frac{\alpha}{4} - \frac{1}{1+M}\frac{\mathrm{d}(1+M)}{\mathrm{d}x} - \gamma\frac{\mathrm{d}M}{\mathrm{d}x} + \frac{\gamma}{4}\frac{\mathrm{d}M^2}{\mathrm{d}x} + \frac{\gamma^2-1}{3}\frac{\mathrm{d}M^3}{\mathrm{d}x} \tag{7}$$

  It should be noted that the conditions which yield Eq. (5) imply that the above conditions will always be met.

- When $b = k_0/(1-M)$, the simplified solution for a is:

$$a^- = \frac{\alpha}{4} - \frac{1}{1-M}\frac{\mathrm{d}(1-M)}{\mathrm{d}x} + \gamma\frac{\mathrm{d}M}{\mathrm{d}x} + \frac{\gamma}{4}\frac{\mathrm{d}M^2}{\mathrm{d}x} - \frac{\gamma^2-1}{3}\frac{\mathrm{d}M^3}{\mathrm{d}x} \tag{8}$$

Combining the simplified solutions for both a and b yields the overall analytical solution of Eq. (3):

$$\hat{p}(x,\omega) = \mathscr{C}^+\mathscr{P}^+(x,\omega) + \mathscr{C}^-\mathscr{P}^-(x,\omega) \tag{9}$$

where

$$\mathscr{P}^+(x,\omega) = \left(\frac{\bar{\rho}}{\bar{\rho}_1}\right)^{1/4}\frac{1+M_1}{1+M}\frac{\exp\left(\gamma M_1 - \frac{\gamma}{4}M_1^2 - \frac{\gamma^2-1}{3}M_1^3\right)}{\exp\left(\gamma M - \frac{\gamma}{4}M^2 - \frac{\gamma^2-1}{3}M^3\right)}\exp\left(-\mathrm{i}\omega\int_{x_1}^{x}\frac{\mathrm{d}x}{\bar{c}+\bar{u}}\right) \tag{10}$$

$$\mathscr{P}^-(x,\omega) = \left(\frac{\bar{\rho}}{\bar{\rho}_1}\right)^{1/4}\frac{1-M_1}{1-M}\frac{\exp\left(\gamma M + \frac{\gamma}{4}M^2 - \frac{\gamma^2-1}{3}M^3\right)}{\exp\left(\gamma M_1 + \frac{\gamma}{4}M_1^2 - \frac{\gamma^2-1}{3}M_1^3\right)}\exp\left(\mathrm{i}\omega\int_{x_1}^{x}\frac{\mathrm{d}x}{\bar{c}-\bar{u}}\right) \tag{11}$$

Table 1: Parameters used in the analysis. They are fixed unless otherwise stated.

l [m]	x_1 [m]	$\bar{p}_1$ [Pa]	$\overline{T}_1$ [K]	$\overline{T}_2$ [K]	M_1 [-]	γ [-]	R_g [J K $^{-1}$ kg^{-1}]
1	0	1×10^5	1600	800	0.2	1.4	287

$\mathscr{C}^+$ and $\mathscr{C}^-$ are two arbitrary coefficients which can be determined for given initial or boundary conditions and M_1 and $\bar{\rho}_1$ are the Mach number and mean density at $x = x_1$ respectively.

Substituting Eqs. (9)-(11) into Eqs. (1)-(2) and neglecting small terms yield the solution of velocity perturbation:

$$\begin{aligned}\hat{u}(x,\omega) = &\left(\frac{\mathrm{i}k_0 - (1+2(1+\gamma)M + (3\gamma-7)M^2)\,\alpha/4}{\mathrm{i}k_0 - \alpha M}\right)\frac{\mathscr{C}^+\mathscr{P}^+(x,\omega)}{\bar{\rho}\bar{c}} \\ &- \left(\frac{\mathrm{i}k_0 + (1-2(1+\gamma)M + (3\gamma-7)M^2)\,\alpha/4}{\mathrm{i}k_0 - \alpha M}\right)\frac{\mathscr{C}^-\mathscr{P}^-(x,\omega)}{\bar{\rho}\bar{c}}\end{aligned} \tag{12}$$

4. Validation duct flows and definition of transfer functions

The accuracy of the analytical solution proposed in Eqs. (9) - (11) and (12) is evaluated by applying it to a straight duct containing one-dimensional mean flow, and with a distributed mean temperature zone extending from $x = x_1$ to $x = x_2 = x_1 + l$, as sketched in Fig. 1. The duct is open at the inlet and the outlet is an anechoic boundary. Two actuators are placed at the entrance of the duct to provide external forcing pressure waves $\hat{p}_e$, resulting that the inlet boundary condition is $\hat{p}(x_1) = \hat{p}_e$.

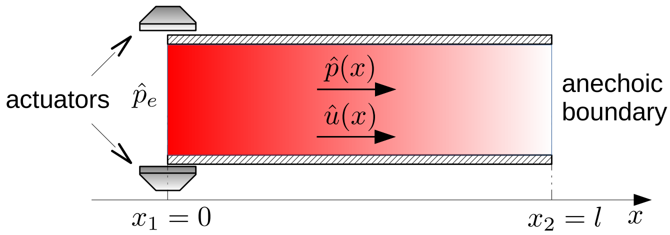

Figure 1: Sketch of the 1-D duct. The actuator is at the inlet of the duct.

Two mean temperature profiles are considered, the first being a linear profile, satisfying:

$$\overline{T}(x) = \overline{T}_1 + \kappa(x - x_1), \quad \text{where} \quad \kappa = (\overline{T}_2 - \overline{T}_1)/l \quad \text{and} \quad x \in [x_1, x_1 + l], \tag{13}$$

The second temperature profile is a more complicated sine wave satisfying:

$$\overline{T}(x) = \frac{\overline{T}_1 - \overline{T}_2}{2}\sin\left(\frac{5\pi}{4}\frac{x - x_1}{l} + \frac{\pi}{4}\right) + \frac{\overline{T}_1 + \overline{T}_2}{2}, \quad \text{where} \quad x \in [x_1, x_1 + l]. \tag{14}$$

Although asymptotic expansions, such as the Magnus expansion, can successfully solve the quasi-one-dimensional LEEs (e.g., Eqs. (1) - (2)) with arbitrary mean parameters profiles [13], there has been no exact or approximate analytical solution represented as the superposition of waves travelling in either direction which is precise enough for this kind of profile until now. The parameters used for the analysis in the following sections are listed in Table 1 unless otherwise stated. It is thereby possible to define transfer functions from $\hat{p}_e(f)$ to $\hat{p}(x, f)$ and $\hat{u}(x, f)$ for an arbitrary location, x, within the temperature profile region, $x_1 \leq x \leq x_2$, and arbitrary frequency, f:

$$\mathcal{F}_p(x, f) = \hat{p}(x, f)/\hat{p}_e(f) \quad \text{and} \quad \mathcal{F}_u(x, f) = \bar{\rho}_1\bar{c}_1\hat{u}(x, f)/\hat{p}_e(f) \tag{15}$$

where $\bar{\rho}_1$ and $\bar{c}_1$ are the mean density and speed of sound at the entrance of the duct respectively. The performance of the analytical solution can be evaluated by comparing the predicted pressure and velocity transfer functions, $\mathcal{F}_p(x, f)$ and $\mathcal{F}_u(x, f)$ respectively, to those numerically calculated using the two linearised Euler equations (2 LEEs method) in pressure and velocity perturbation defined in

Eqs. (1) and (2). For the 2 LEEs method, spatial discretisation is via a second order finite difference scheme on a uniform grid containing 5×10^3 points. The difference between the predicted and LEE calculated transfer functions is quantified using an error coefficient, defined as:

$$\epsilon = \frac{1}{2}\Big[\Big(\frac{1}{N}\sum_{j=1}^{N}|\mathcal{F}_p(x_j) - \mathcal{F}_{p,\mathrm{LEE}}(x_j)|^2\Big)^{1/2} + \Big(\frac{1}{N}\sum_{j=1}^{N}|\mathcal{F}_u(x_j) - \mathcal{F}_{u,\mathrm{LEE}}(x_j)|^2\Big)^{1/2}\Big], \quad x_j = x_1, x_2, \cdots x_N \tag{16}$$

5. Results and discussion

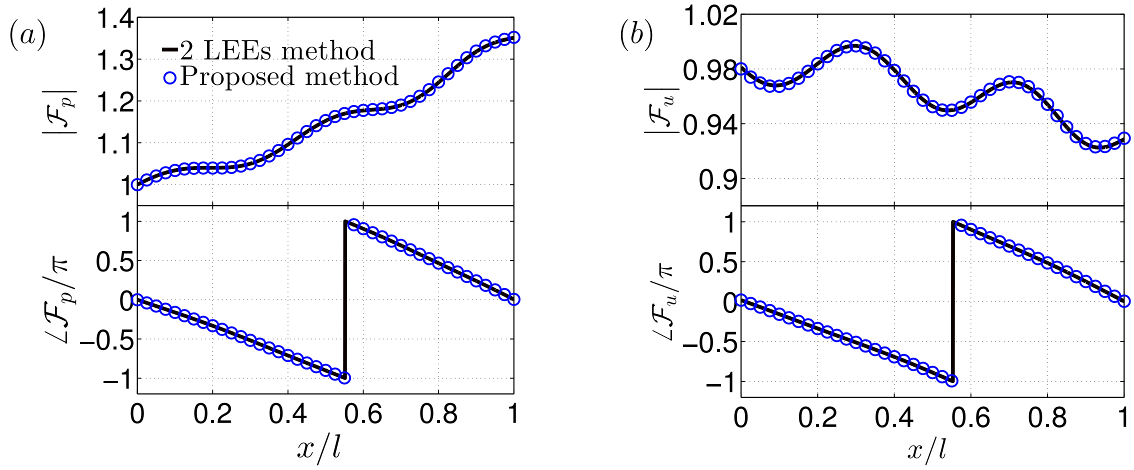

Figure 2: Evolution of $\mathcal{F}_p$ and $\mathcal{F}_u$ with x for the linear mean temperature profile. $fl/\bar{c}_1 = 1$. (a): $\mathcal{F}_p$. (b): $\mathcal{F}_u$.

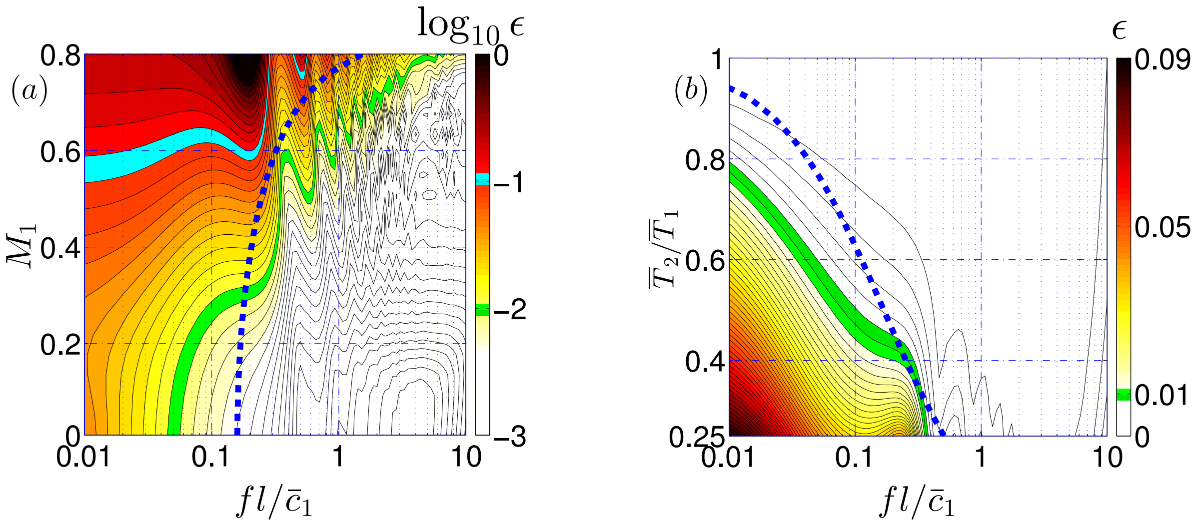

Figure 3: (a) Contour map of ϵ as functions of $fl/\bar{c}_1$ and M_1. (b) Contour map of ϵ as functions of $fl/\bar{c}_1$ and $\overline{T}_2/\overline{T}_1$. - - - represents χ.

Predictions from the proposed analytical solution are compared to those using the LEEs. Figure 2 shows the predicted gain and phase of the acoustic transfer functions against axial distance in the case of a linear mean temperature distribution and a moderate Mach number mean flow ($M_1 = 0.2$). The other parameters are $\overline{T}_1 = 1600$ K, $\overline{T}_2 = 800$ K, $l = 1$ m and $fl/\bar{c}_1 = 1$. Predictions using the present analytical solution show excellent agreement with those calculated using the LEEs.

A parametric study is now performed to investigate the accuracy of the proposed analytical solution across different frequencies, $fl/\bar{c}_1$, Mach numbers, M_1 and mean temperature gradients (changing $\overline{T}_2/\overline{T}_1$ herein). Figure 3(a) shows the error coefficients, ϵ, as functions of $fl/\bar{c}_1$ and M_1, using the proposed solution. The acoustic field is captured accurately, even for large inlet Mach numbers of $M_1 = 0.6$. Furthermore, the proposed analytical solution works well at low frequencies, even though the analytical solution made the assumption $|k_0| \gg |\alpha|$ and $|k_0^2| \gg |\beta|/2$. For the linear mean temperature profile, $\beta \equiv 0$, and the frequency condition becomes:

$$\frac{fl}{\bar{c}_1} \gg \chi = \frac{l}{2\pi\bar{c}_1}\max\left(|\alpha|\bar{c}\right) = \frac{l}{2\pi\bar{c}_1}\max\left|\frac{\bar{c}}{1-\gamma M^2}\frac{1}{\overline{\overline{T}}}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right| \tag{17}$$

As shown in Fig. 3(a), when $fl/\bar{c}_1 > \chi$ or $|k_0| > |\alpha|$, high accuracy is always guaranteed even for large Mach number. It should be noted here that the normalised first resonance frequency of the duct is $fl/\bar{c}_1 = 0.5$ when both ends are open to atmosphere and there is no mean temperature gradient nor mean flow. Thus the typical acoustic duct frequencies relevant to thermoacoustic instability are "large" according to the above definitions, and the proposed analytical solution would be suitable for their analysis. The error coefficient ϵ as functions of $fl/\bar{c}_1$ and $\overline{T}_2/\overline{T}_1$, are shown in Figure 3(b). As $\overline{T}_2/\overline{T}_1$ and $fl/\bar{c}_1$ decrease, the errors increase and the predicted acoustic field loses accuracy. However, when $fl/\bar{c} > \chi$ or $|k_0| > |\alpha|$, an error of less than 1% is always achieved. It is furthermore noted that this method can be applied to large mean temperature gradients.

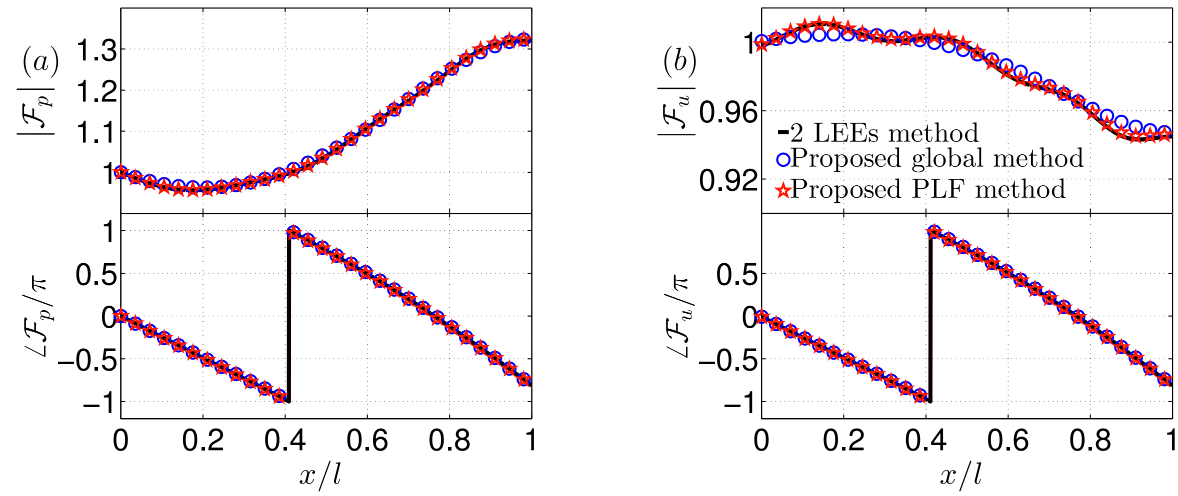

Figure 4: Evolution of $\mathcal{F}_p$ and $\mathcal{F}_u$ with x for the sine mean temperature profile. $fl/\bar{c}_1 = 1.5$. (a): $\mathcal{F}_p$. (b): $\mathcal{F}_u$.

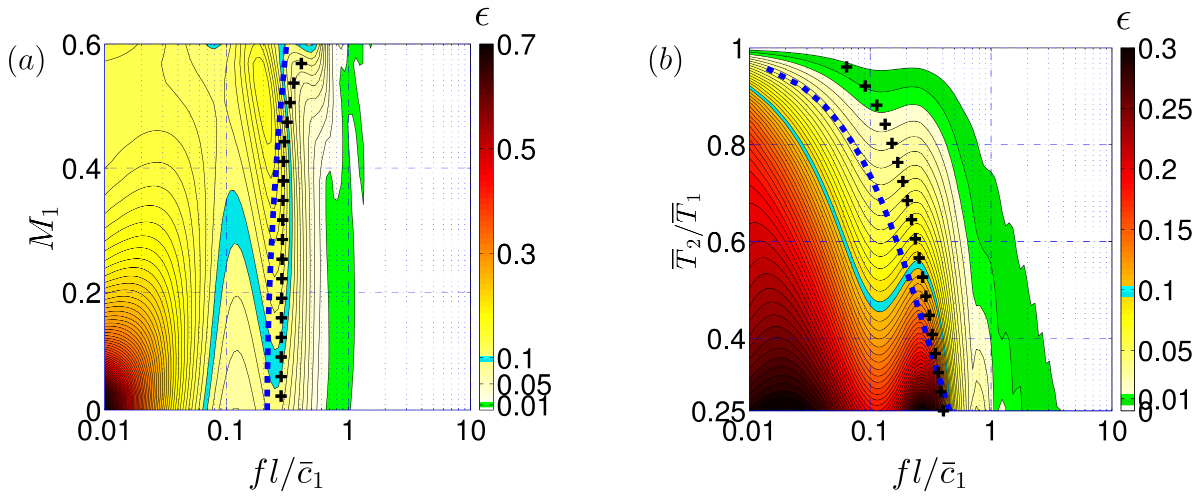

Figure 5: Proposed analytical solution applied globally. (a) Contour maps of ϵ as functions of $fl/\bar{c}_1$ and M_1. (b) Contour maps of ϵ as functions of $fl/\bar{c}_1$ and $\overline{T}_2/\overline{T}_1$. - - - : χ. + : ς.

For the axial mean temperature variation which has a sine-wave profile, there has previously been no analytical solution represented as the superposition of waves travelling in either direction, even when there is no mean flow. The above part showed that the proposed method performs very well for linear mean temperature profiles. The proposed method is therefore applied using two approaches: (i) directly to the global temperature field and (ii) using a piecewise linear approximation to the temperature field. For the latter, the axial length over which the temperature change occurs is segmented

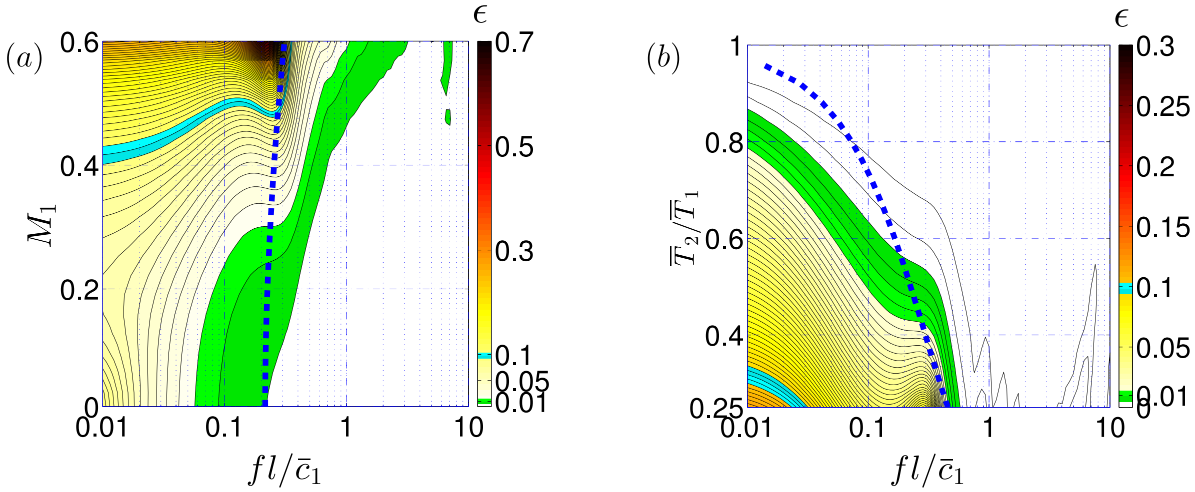

Figure 6: Proposed analytical solution applied in piecewise linear form. (a) Contour maps of ϵ as functions of $fl/\bar{c}_1$ and M_1. (b) Contour maps of ϵ as functions of $fl/\bar{c}_1$ and $\overline{T}_2/\overline{T}_1$. - - - : χ. $N_s = 20$.

into N_s equal lengths. Linear least-squares fitting is applied to the temperature profile within each segment to get a piecewise linear function (PLF). The proposed analytical solution is then applied separately to each segment containing a different linear temperature profile.

The predicted acoustic transfer functions and their comparison to calculations from the LEEs are shown in Figure 4. The thermodynamic and flow properties are: $\overline{T}_1 = 1600$ K, $\overline{T}_2 = 800$ K, $M_1 = 0.2$ and $fl/\bar{c}_1 = 1.5$. The number of axial segments used is $N_s = 20$. The proposed analytical solution, when applied globally to the temperature field, is not able to reproduce the acoustic field precisely. However, when applied using the piecewise linear approach, accurate predictions of the acoustic field are achieved. For the sine wave mean temperature profile, the effect of the second order spatial gradient of mean density, β, cannot be neglected. As well as the condition given in Eq. (17), the following condition should also be satisfied:

$$\frac{fl}{\bar{c}_1} \gg \varsigma = \frac{l}{2\pi\bar{c}_1} \max\left(\bar{c}\left(\frac{|\beta|}{2}\right)^{1/2}\right) \tag{18}$$

where

$$\beta = \frac{2(\gamma^2 M^4 - \gamma M^2 + 1)}{(1-\gamma M^2)^3}\left(\frac{1}{\overline{T}}\frac{\mathrm{d}\overline{T}}{\mathrm{d}x}\right)^2 - \frac{1}{1-\gamma M^2}\frac{1}{\overline{T}}\frac{\mathrm{d}^2\overline{T}}{\mathrm{d}x^2} \tag{19}$$

A parametric study is performed to investigate the accuracies of the two approaches across different frequencies, $fl/\bar{c}_1$, Mach numbers, M_1 and mean temperature gradients (characterised by $\overline{T}_2/\overline{T}_1$). The error coefficient as a function of $fl/\bar{c}_1$ and M_1, are shown in Figures 5(a) and 6(a) for the proposed analytical solution applied globally and in piecewise linear form respectively. Unlike for linear mean temperature profiles, the proposed solution applied globally is only accurate when $fl/\bar{c}_1 \gg \chi$ and $fl/\bar{c}_1 \gg \varsigma$ are satisfied. However, when applied in piecewise linear form, as long as $fl/\bar{c}_1 > \chi$ or $|k_0| > |\alpha|$, good accuracy is always achieved even for very large Mach numbers e.g. $M_1 = 0.6$. Figures 5(b) and 6(b) show the error coefficient ϵ as functions of $fl/\bar{c}_1$ and $\overline{T}_2/\overline{T}_1$. The same trend is found.

6. Conclusions

The present work has presented an analytical solution for the one-dimensional acoustic field in a duct with an arbitrary axial mean temperature gradient and mean flow. The proposed analytical solution has been applied to two axial mean temperature profiles, the first being linear and the second having a partial sine wave variation. Results were compared to those numerically calculated using the linearised Euler equations. For the linear mean temperature profile, as long as $|k_0| > |\alpha|$ (where k_0 is the local wave number when there is no mean flow and α is the normalised mean density gradient), the proposed analytical solution captures the acoustic field very accurately, even for moderate-to-large subsonic Mach numbers. For the sine wave mean temperature profile, the proposed analytical solution directly applied to this profile works well for large enough frequencies. However, errors occur at low frequencies. These can be overcome by applying the proposed analytical solution to an axially segmented representation of the duct, each approximated to contain a linear mean temperature profile. Results show that this piecewise linear method can capture the acoustic field very accurately, even for low frequencies, enabling the use of the proposed analytical solution to an arbitrary mean temperature profile.

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