PRACTICAL BOUNDS FOR THE ANGULAR RESOLVING-POWER

J. Munier and J.L. Lacoume

Centre d'Etude des Phénomènes Aléatoires et Géophysiques
BP 46, 38402 St. Martin d'Hères Cedex - France -

1 - Preliminary comments.

The resolving power undoubtedly is a major performance criterion for spatial or spectral analysis methods. This paper does not pretend to give a definitive answer to this question, but rather to bring to light its intricacy. Owing to the variety of phenomena involved, several criteria will be suggested, one of them being possibly the most suitable according to the circumstances. As an initial hypothesis, the cross-spectral matrix computed from the output signals of the receiving array is assumed to have been estimated without noticeable errors. Therefore, only the problems to be due to the structure of the source field and to the fluctuating medium will be considered. These latter, indeed, constitute the ultimate causes of limitation.

It is well known that at most n-1 sources can be identified with a n-sensors array via a parametric method (typically the Pisarenko method in the case of plane waves and a linear array of identical and equispaced sensors [1]). Consequently, all the extra-sources (generally the weakest sources) must be incorporated to the background noise. In most cases, these sources constitute the main part of the noise : it may be observed that they correspond possibly to multipaths coming from some of the main sources.

Considering the problem of the resolving power, only the two sources to be separated are now to be classified as "signals" and all the other sources and jammers to be classified as "noise". Observing that three channels are sufficient to identify two sources, it is obviously advisable to group together the sensors into three (directive) subarrays, so that the signals coming from the two sources of interest be predominant. This question will be further developped through numerical simultations on a typical example.

Also one must recall that all the continuous analysis methods (Capon, MEM-AR, Borgiotti-Lagunas, MUSIC) asymptotically exhibit the same resolving power as the Pisarenko method [2], which is then to be taken as a reference.

2 - Usefulness of spatial prefiltering and necessity of pre-orientating.

A typical situation that may be encountered in aerial or underwater acoustics is shown by fig.1, where it can be seen five dominant sources and a lot of weak sources considered as jammers. Applying the Pisarenko method to the 8x8 cross-spectral matrix observed at the output of a eight-sensors linear array yields the seven solutions visible on the figure. It is to be noticed that the two closely spaced dominant sources (a) and (b) are not resolved and further that the spacings between the strongest solutions are large enough to permit the application of a spatial filtering technique.

With a 2n-sensors array, three overlapped n-sensors subarrays (for example) may be formed ; for the above-mentioned 8-sensors array, the subarrays respectively group the sensors 1 to 4, 3 to 6, 5 to 8 ; the weighting for a subarray may be 0.5, 1, 1, 0.5, which gives a 2x42 degrees beamwidth (between nulls) with a low level (-24 dB) secondary lobe.

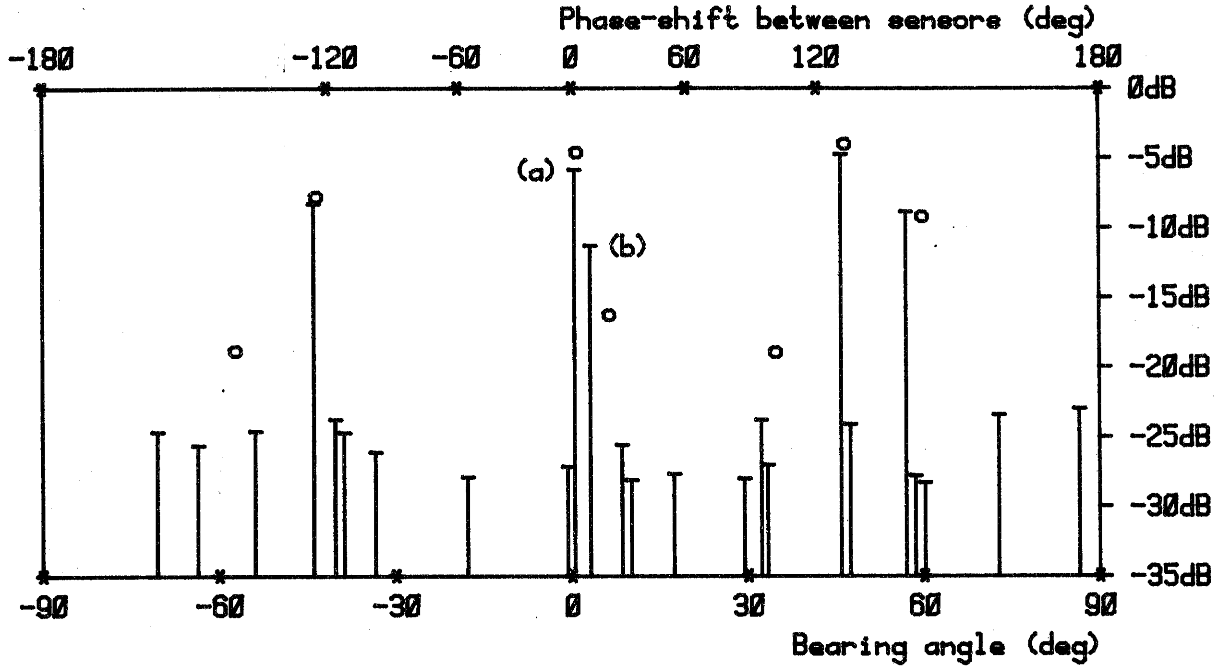

Fig. 1- A typical case : 5 sources + 19 jammers : 8 equispaced sensors
——| True sources and jammers (0 dB level : sum of intensities)
o PISARENKO solutions (0 dB level : sum of estimated intensities)

In order to examine if some parametric solution or other corresponds either to a single source or to two closely spaced sources, the whole array must be suitably directed (pre-oriented) by means of phase-shifts applied to the sensors outputs. To the question "What is the suitable pre-orientation ?", the answer is given by fig. 2. A three channels system always yields two parametric solutions but, depending on the amount of perturbation due to parasitic sources and jammers and then depending on the pre-orientation, these solutions may vary considerably, as seen on the figure. From the observation of numerous cases, it could be concluded that the presence of two sources (rather than one) almost surely is ascertained if the estimated angular spacing locally is nearly stable and if simultaneously the estimated intensities are significantly great.

Moreover, the pre-orientation that yields a minimal angular separation gives the better estimates and further corresponds roughly to a maximum of the 2nd eigenvalue to 3rd eigenvalue ratio. These worthwhile properties though experimentally verified have not been theoretically proved to date.

For the case of fig.2, the optimal pre-orientation appears to be equal to 0.2 deg., a value which gives the final result shown by fig. 3. The resolution of the sources (a) and (b) is satisfactorily achieved, this being due to the fact that the parasitic sources and jammers have been largely eliminated.

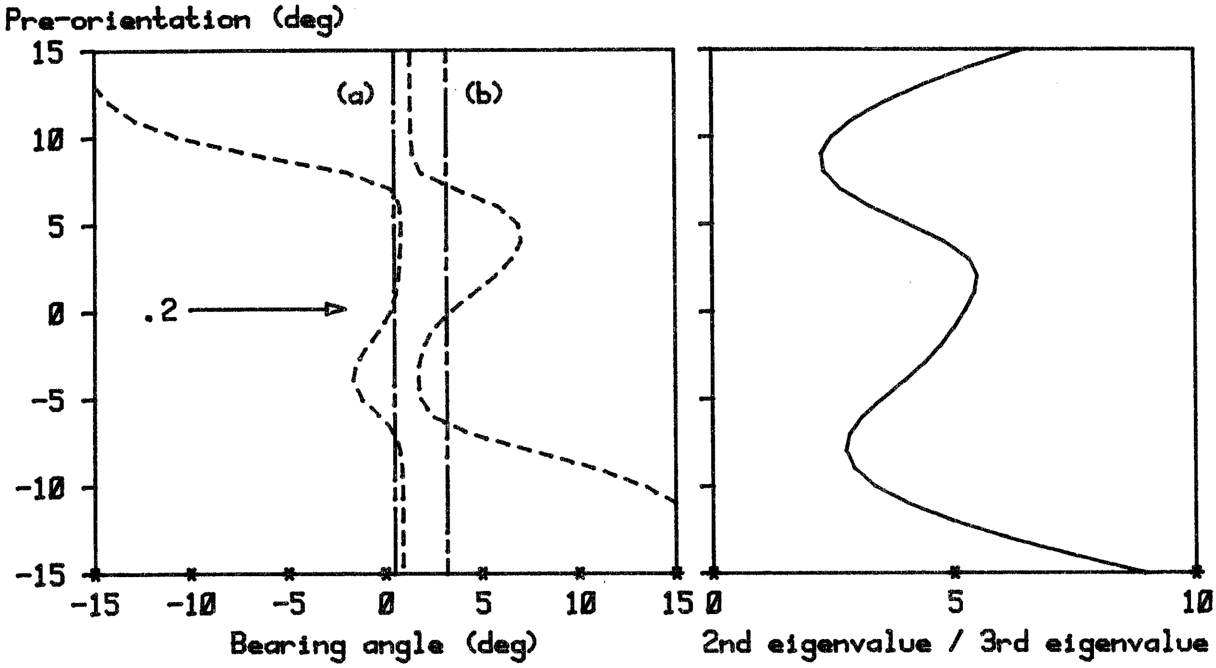

Fig. 2- Direction finding with a 8 sensors array divided into 3 overlapped 4 sensors subarrays. Data of fig. 1. Influence of the array pre-orientation

———— True bearings ‐ ‐ ‐ ‐ ‐ ‐ ‐ ‐ Estimated bearings

3 -<u>A first criterion based upon the cross-spectral matrix eigenvalues</u>.

In the case of spectral analysis, the covariance matrix eigenvalues are known to be useful to conceive criteria for determining the number of spectral lines (e.g. Akaike, Rissanen [3]). In the spatial case, additional phenomena occur, owing to the turbulence of the medium and the variability of the reverberation. These latter are to be considered as generating multiplicative (or modulating) rather than additive noise effects. The influence of "true" additive noise components, mainly due to parasitic sources, and their reduction via spatial prefiltering were examined in the last section.

In order to evaluate the interest of using eigenvalues based criteria in the spatial case, it is now advisable to introduce some propagation effects in the simulation. This was done using a new situation depicted by fig. 4. The direction-finding process (by means of the same antenna system as in fig. 2 and 3) is here applied to the case of equal intensities sources crossing each other in the presence of a lot of jammers. The SNR for one source before spatial filtering was 1.43 dB. The pre-orientation of the array was optimally adjusted for the two sources merged at zero bearing. The dashed lines indicate the parametric solutions and eigenvalues computed in the absence of propagation perturbations. It can be seen that the angular sources spacing is ill-estimated when less than about 3 degrees, a value to be compared to the 14.5 degrees Rayleigh limit for the 8 equispaced ($\lambda/2$ apart) sensors array here considered. Furthermore, the limit of resolution corresponds to a 2nd eigenvalue to 3rd eigenvalue ratio roughly equal to 3.

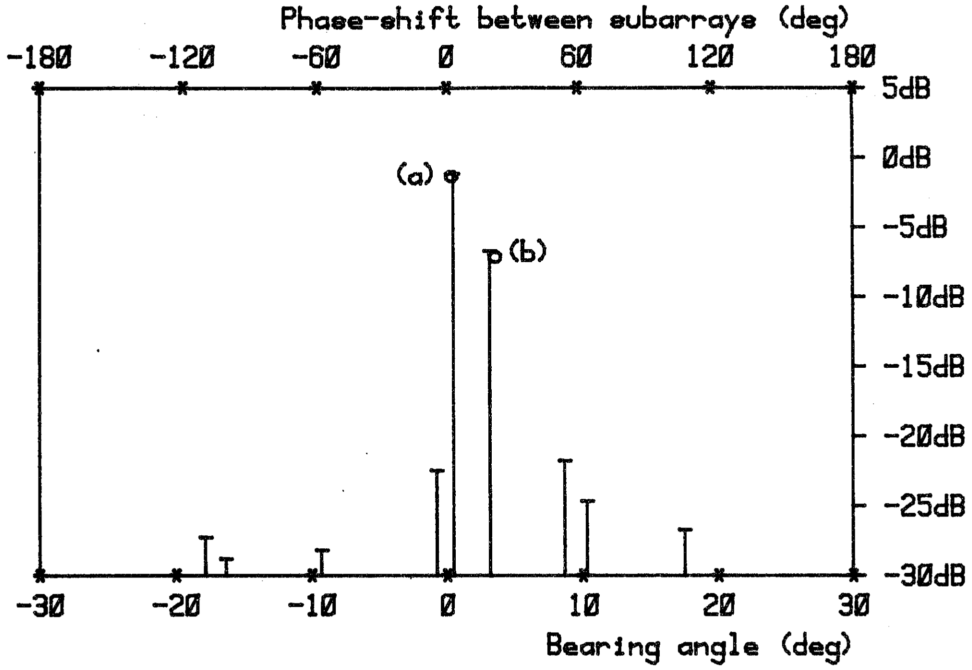

Fig. 3- Resolution with a .2 deg. array pre-orientation (cf. fig. 2)
——————┤ True sources and jammers as seen through the directive 4 sensors subarrays o PISARENKO solutions

The fluctuating effects (especially resulting from the turbulence of the medium and from moving reflectors) are too much complex to be examined in detail here. A simple example is given by fig. 4 ; for all the sources and jammers, statistically independent fluctuations have been introduced : a) for the angles of incidence, taken with gaussian distribution having a 0.3 deg. standard deviation ; b) for the intensities, taken with a distribution having a 50% relative standard deviation and a mean value equal to the one observed without fluctuations. It appears that the estimation results (solid line drawn), in spite of noteworthy eigenvalues fluctuations, are not much worse than in the quiet case. The limit of resolution may be estimated equal to about 5 degrees ; it corresponds to a mean value of the 2nd to 3rd eigenvalues ratio equal to about 5.

From the preceding examples, it may be concluded that the 2nd to 3rd eigenvalues ratio is an acceptable performance index for testing hypothesis "two sources vs a single source" ; nevertheless, the detection threshold greatly depends upon the nature of perturbations. If the noise matrix (including any jammer or parasitic source) nearly have the $\sigma^2 \mathbf{I}$ form, then the threshold can be taken a little greater than unity ; values of 3 to 5 appear to be more realistic under current circumstances.

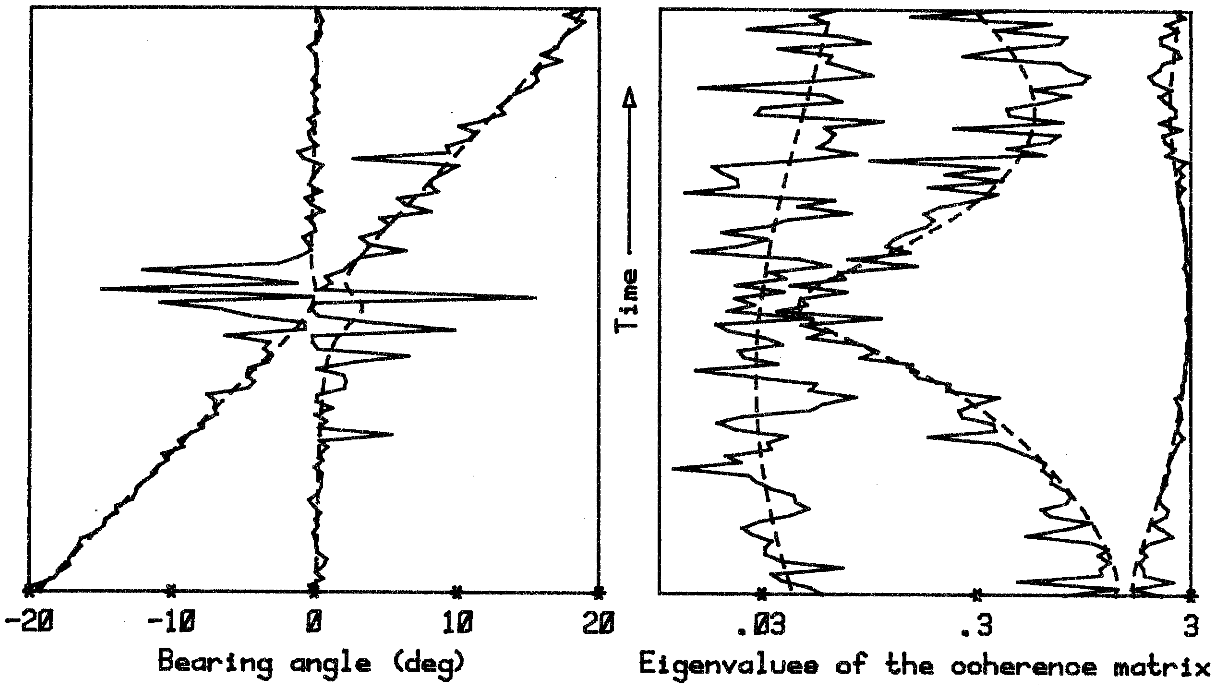

Fig. 4- Resolution of 2 sources crossing each other
Same jammers and antenna system as in fig. 2 and 3
———— with turbulence of the medium - - - - - - without

In order to obtain a straightforward algebraic expression of the resolving power, the simple case of two equal intensities uncorrelated sources in the presence of "white noise" may be considered. In such a case, the cross-spectral matrix of the signals observed at the outputs of a n equispaced sensors array with uniform weighting is as :

$$\mathbf{\Gamma} = \mu \, (\mathbf{u}_a \mathbf{u}_a^+ + \mathbf{u}_b \mathbf{u}_b^+) + \sigma^2 \mathbf{I} \quad , \tag{1}$$

the real scalar quantity μ being the intensity of the sources and $\mathbf{u}_a$, $\mathbf{u}_b$ their unit directional vectors, the general form of which is, for plane waves and equispaced sensors :

$$\mathbf{u} = 1/\sqrt{n} \, [1, e^{-j\varphi} , \ldots e^{-j(n-1)\varphi}]^T \quad , \tag{2}$$

the quantity φ being the phase-shift between sensors, expressed as :

$$\varphi = 2\pi \frac{d}{\lambda} \sin \theta \quad , \tag{3}$$

(λ: wavelength, d: spacing between sensors, θ: bearing angle referred to broadside).

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For the case of two sources (a) and (b) to be separated, the quantities of interest are :

$$\begin{cases} \tau = \theta_a - \theta_b \\ \delta = \varphi_a - \varphi_b \xrightarrow[\tau \to 0]{} 2\pi \dfrac{d}{\lambda} \tau \cos \dfrac{\theta_a + \theta_b}{2} \end{cases} \quad . \tag{4}$$

It can be shown that the eigenvalues of the nxn matrix $\mathbf{\Gamma}$ express as :

$$\begin{cases} \lambda_1 = \mu \left[1 + \dfrac{\sin(n\delta/2)}{n \sin(\delta/2)} \right] + \sigma^2 \\ \lambda_2 = \mu \left[1 - \dfrac{\sin(n\delta/2)}{n \sin(\delta/2)} \right] + \sigma^2 \\ \lambda_i = \sigma^2 \quad , \quad i > 2 \end{cases} \tag{5}$$

The hypothesis "two sources" is accepted if the following condition is verified :

$$\frac{\lambda_2}{\lambda_i(i>2)} \geq K_1 \tag{6}$$

K_1 being the detection theshold.

Using limited expansion for the sine, the resolving power, defined by the equality in the conditions (6), can be found to be :

$$\delta_0 (n) = \left[\frac{24\ (K_1 - 1)}{n(n^2-1)\ R} \right]^{1/2} \tag{7}$$

R being the SNR defined as :

$$R = \frac{\mu}{n\ \sigma^2} \quad . \tag{8}$$

The angular resolving power then expresses as, in the limit,

$$\tau_0 (n) \xrightarrow[R \to \infty]{} \frac{1}{2\pi \dfrac{d}{\lambda} \cos \dfrac{\theta_a + \theta_b}{2}} \left[\frac{24\ (K_1 - 1)}{n(n^2-1)\ R} \right]^{1/2} \quad \text{(radian)} \quad . \tag{9}$$

If n sensors are assembled into three n/2 sensors subarrays (see sections 2 and 3), the interferometric spacing between subarrays is nd/4. Therefore, the angular resolving power for near-broadside directions is given by :

$$\tau_0(3) \xrightarrow[R\to\infty]{} \frac{2\lambda}{\pi n d}\left[\frac{K_1-1}{R}\right]^{1/2} \quad \text{(radian)}, \qquad (10)$$

hence :

$$\frac{\tau_0(3)}{\tau_0(n)} = 2\left[\frac{n^2-1}{6n}\right]^{1/2} . \qquad (11)$$

It appears to be more efficient to process directly the n x n matrix rather than the 3 x 3 matrix obtained with prefiltering. This is true with the "white noise" hypothesis. In the presence of parasitic sources, it is better to apply a prefiltering procedure, as it was previously shown.

4 - A second criterion based upon the eigenvalues variances.
In many cases, the variability of the eigenvalues is the dominant factor of limitation, for it corresponds to fluctuations of the estimated bearings. As a practical rule, it may be considered that the limit of separability for two sources is attained when the relative standard deviation (i.e. the standard deviation divided by the mean value) of their estimated angular spacing $\hat{\tau}$ is equal to a certain constant smaller than unity. For sources near broadside, this condition may be expressed, using the estimated differential phase-shift $\hat{\delta}$, as : for $K_2 > 1$

$$\frac{(\mathrm{var}\{\hat{\delta}\})^{\frac{1}{2}}}{E\{\hat{\delta}\}} = \frac{1}{K_2} \qquad (12)$$

where $E\{\hat{\delta}\}$ may be replaced by δ in a first approximation. For the case of three subarrays, which is of major interest to separate closely spaced sources, as seen in the preceding sections, the formulas (5) can be reduced to :

$$\begin{cases} \lambda_1 = \dfrac{2\mu}{3}(2+\cos\delta)+\sigma^2 \\ \lambda_2 = \dfrac{2\mu}{3}(1-\cos\delta)+\sigma^2 \\ \lambda_3 = \sigma^2 \end{cases} \qquad (13)$$

From this, an estimation of the differential phase-shift δ can be obtained. Using a limited expansion for the cosine gives :

$$\delta^2 = 6 \frac{\lambda_2 - \lambda_3}{1 + \lambda_2 - 2\lambda_3} \quad . \tag{14}$$

A further simplification occurs if using coherence matrix, rather than the cross-spectral matrix, the the sum of its eigenvalues being equal to 3 ; then :

$$\delta^2 = 2 \frac{\lambda_2 - \lambda_3}{1 - \lambda_3} \quad . \tag{15}$$

Assuming constant intensities for the two main sources and small fluctuations, so that the formulas (13) nearly remain valid, the following relationship can be derived :

$$\delta^2 \text{ var } \{\hat{\delta}\} = \text{var} \left\{ \frac{\lambda_2 - \lambda_3}{1 - \lambda_3} \right\} \simeq \text{var } \{\lambda_2 - \lambda_3\} \quad . \tag{16}$$

Thus, using (12), the resolving power can be characterized by :

$$\delta_0 = (K_2 \, \sigma_\Lambda)^{\frac{1}{2}} \qquad \text{(radian)}, \tag{17}$$

where : $\sigma_\Lambda = [\text{ var } \{\lambda_2 - \lambda_3\} \,]^{\frac{1}{2}}$ is the standard deviation of $\lambda_2 - \lambda_3$.

Applying this to the n-sensors array divided into three n/2-sensors subarrays described in section 2 gives for the angular resolving power :

$$\boxed{\tau_0\,(3) = \frac{2\lambda}{\pi n d}\,(K_2 \sigma_\Lambda)^{\frac{1}{2}}} \qquad \text{(radian)}. \tag{19}$$

This result may be compared to the expression (10), which is to be rewritten in the case of a 3 x 3 coherence matrix. From the definition (8) and the property $\lambda_1 + \lambda_2 + \lambda_3 = 3$, it is found that :

$$R = \frac{1}{2} \left[\frac{1}{\lambda_3} - 1 \right] \simeq \frac{1}{2\lambda_3} \quad . \tag{20}$$

Thus (10) becomes :

$$\boxed{\tau_0\,(3) = \frac{2\lambda}{\pi n d}\,[\, 2\lambda_3 (K_1 - 1)]^{\frac{1}{2}}} \qquad \text{(radian)}. \tag{21}$$

For the situation of fig. 4, with $d/\lambda = 0.5$, $n = 8$, $\lambda_3 = 0.03$ and $K_1 = 3$, this gives $\tau_0 = 3.2$ degrees. Such a value concerns the case without fluctuations. Under the fluctuating conditions of fig. 4, the standard deviation of $\lambda_2 - \lambda_3$ is found to be about 0.035. Then applying the formula (19) gives $\tau_0 = 3.8$ degrees for $K_2 = 5$, a result that roughly agrees with fig. 4.

5 - <u>A third criterion based on the variance of the cross-spectral matrix elements.</u> For two equal intensities sources (a) and (b), the cross-spectral matrix in the absence of noise is :

$$\mathbf{\Gamma} = \mu\,(\mathbf{u}_a\,\mathbf{u}_a^+ + \mathbf{u}_b\,\mathbf{u}_b^+) \quad , \qquad (22)$$

the unit directional vectors $\mathbf{u}_a$ and $\mathbf{u}_b$ being given by (2).

The total received power may be characterized by (not equal to) the euclidian norm squared of $\mathbf{\Gamma}$, that is :

$$\|\mathbf{\Gamma}\|^2 = \frac{1}{n}\,\mathrm{tr}\,(\mathbf{\Gamma}^+\mathbf{\Gamma}) \quad . \qquad (23)$$

Noting that the scalar product $\mathbf{u}_a\,\mathbf{u}_b^+$ is nearly equal to unity in the case of closely spaced sources, it can be easily shown that :

$$\|\mathbf{\Gamma}\|^2 = \frac{4\,\mu^2}{n} \quad . \qquad (24)$$

In presence of noise, jammers or other perturbations, the measured cross-spectral matrix may be expressed as :

$$\hat{\mathbf{\Gamma}} = \mathbf{\Gamma} + \tilde{\mathbf{\Gamma}} \quad , \qquad (25)$$

where $\tilde{\mathbf{\Gamma}}$ is the perturbation matrix, the elements of which being assumed to have zero mean value. The perturbation power may be characterized by the quantity :

$$w = E\{\|\tilde{\mathbf{\Gamma}}\|^2\} = E\{1/n \sum_i \sum_j |\tilde{\Gamma}_{ij}|^2\} \quad . \qquad (26)$$

Assuming that the elements of the matrix $\tilde{\Gamma}$ have the same variance v (which is in most cases verified) defined by :

$$v = E\{|\tilde{\Gamma}_{ij}|^2\} \quad , \forall_{i,j} \quad , \qquad (27)$$

then $w = nv$. (28)

It is important to remark that the cross-spectral matrix is here taken as the true observation, rather than the sensors output signals themselves. Therefore, a "special" SNR may be defined for one source as :

$$\rho = \frac{1/2\,\|\Gamma\|^2}{E\{\|\tilde{\Gamma}\|^2\}} = \frac{2\,\mu^2}{n^2\,v} \quad . \tag{29}$$

This quantity exactly corresponds to the quantity R appearing in reference [4], where the resolving power is defined by the condition (12) with $K_2 = 1$, that is : the relative standard deviation of the estimated differential phase-shift $\hat{\delta}$ being equal to unity. In reference [4], the Cramer-Rao bound for the variance of $\hat{\delta}$ is used to calculate the angular resolving power, then resulting in :

$$\tau_0\,(n) = \frac{\lambda}{\pi d}\left[\frac{45}{n^2\,(n^2-1)\,(7n^2-13)\rho}\right]^{1/4} \quad \text{(radian)} \ . \tag{30}$$

In the case of the three subarrays system presented in the preceding sections and in the conditions of fig. 4, the SNR as defined in (29) was found to be equal to 5.3 ; with n = 3 and $d/\lambda = 1$, this gives $\tau_0 = 4$ degrees. This is in accordance with the previous results.

CONCLUSION

Resolving power is like the Lochness Monster, an attractive hundred-years-old subject for conversation. Though some people think to have it recognized, it really remains a quite elusive thing. It exhibits some different "solutions", but one would like to be sure that they could be linked together. There is most probably a lot of works to be undertaken for finding out its hidden parts. A topic where underwater propagation effects play an outstanding role indeed.

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