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Parametric Acoustic Arrays

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Abstract

This paper is a contribution to the theory of parametric acoustic arrays formed by nonlinear interaction of two coaxial, bounded soundbeams. Using perturbation methods, we derive some simplified equations governing the generated sound field, taking into account diffraction, nonlinearity and dissipation. Each equation has a limited range of validity, but the solutions are matched to give a solution valid in the whole space of propagation.

We consider the case when the source of the interacting (primary) waves is a circular piston. The nearfields of the primary waves are approximated by well collimated plane waves, or by Gaussian function shaped beams, whereas the Bessel function directivities of the farfields are approximated by Gaussian function directivities. The effect of oscillations in phase and amplitude in the transient region between the near- and farfields is also analyzed. In a very simple way, the theory accounts for the variation in the propagation curves and beam patterns of the generated sound.

We also consider briefly the case of two infinite plane waves. For strong waves, a simple (Fay-type) solution is obtained for the farfield of a parametric array, based on the exact solution of Burgers' equation.

1. Linearized Soundfield

First, we consider the linearized soundfield. The source is a vibrating circular piston of radius a, mounted in an infinite rigid wall. The normal velocity of the piston is $\mathrm{Re}\left(V_{o1}e^{-i\omega_1 t}\right)$, and the Rayleigh distance is denoted $r_1 = k_1a^2/2$, where $k_1 = \omega_1/c_o$ is the wavenumber. Absorption is ignored for the moment.

It is well known, for instance, from a very thorough numerical analysis by Hobaek[1], that the sound is radiated as a well defined beam in the vicinity of the source, with only small, regular fluctuations in amplitude and phase angle across the beam. It is therefore a good model to assume a plane collimated beam in this region [range $r \lesssim O(r_1/2\pi)$]. However, the beam becomes more diffused and has larger fluctuations in amplitude and phase angle as it is propagated away from the source. Yet, if $k_1r_1 \gg 1$, a simplified analytical expression for the sound pressure can be obtained in this important transient region of the beam[2]. The beam is assumed to be along the z-axis, and the following scaling is introduced, $\sigma = z/r$, $\xi = (x^2+y^2)^{\frac{1}{2}}/a$. A simplified solution can then be found by substituting $P_1 = e^{ik_1z}q_1/z$ in the linearized wave equation and seeking a perturbation solution in q_1, and applying matching

*On leave from the Department of Mathematics, University of Bergen, Bergen, Norway principles. To the leading order in $1/k_1r_1$, we find

$$q_1(\sigma,\xi) = p_o\left(1-e^{\frac{i}{\sigma}}\right)+\frac{2p_o}{\sigma}\int_o^{\xi} e^{\frac{1+t^2}{\sigma}} J_1\left(\frac{2t}{\sigma}\right)dt \quad , \tag{1}$$

which matches an inner, plane wave solution, P^I near the piston $\left(\lim P_1(\sigma \to o) = \lim P^I\ (k_1a \to \infty,\ \sigma \text{ fixed})\right)$. It also leads to the correct asymptotic formula (with Bessel directivity) as $\sigma \to \infty$ (ξ/σ fixed). Here p_o is the pressure amplitude at the piston.

On the axis, we have

$$q_1(\sigma,o) = p_o\left(1-e^{\frac{i}{\sigma}}\right) , \tag{2}$$

with amplitude $2\left|p_o \sin\frac{1}{2\sigma}\right|$ and phase angle, $\phi = \frac{1}{2\sigma}$ + phase ($\sin\frac{1}{2\sigma}$). At the edge of the beam, we have

$$q_1(\sigma,1) = \frac{p_o}{2}\left(1- e^{\frac{2i}{\sigma}} J_o\left(\frac{2}{\sigma}\right)\right) \quad , \tag{3}$$

showing an amplitude with only small fluctuations about $\left|\frac{p_o}{2}\right|$.

The results of Eqs. (2) and (3) are shown in Fig. 1. They are in agreement with the numerical results obtained by Hobaek[1].

This more complete picture of the linearized sound, with changing amplitude and phase angle, is, we believe, important in interpreting many nonlinear effects. The effect of dissipation, so far neglected, can readily be taken into account in these derivations, or simply accounted for by adding an absorption factor in the results.

2. Nonlinear Wave Equation

The basic equation is given by

$$\left(c_o^2 + D\frac{\partial}{\partial t}\right)\nabla^2\rho - \frac{\partial^2\rho}{\partial t^2} = -\nabla\nabla:(\rho\underline{v}\underline{v}) - \nabla\cdot[(c^2-c_o^2)\nabla\rho] \quad , \tag{4}$$

where ρ is the density, $\underline{v}$ the velocity, c and c_o are the isentropic speed of sound at the local and ambient values of pressure and density, respectively, and D is the sound diffusivity (effects of viscosity, heat conduction, relaxation). Further,

$$c^2 = c_o^2 + \left(\frac{d^2P}{d\rho^2}\right)_{\rho=\rho_o}(\rho-\rho_o) + \frac{1}{2}\left(\frac{d^3P}{d\rho^3}\right)_{\rho=\rho_o}(\rho-\rho_o)^2 \tag{5}$$

in this approximation where terms of relative order MS and S^2 are neglected.

(M is the Mach number and S the Stokes number associated with the fundamental modes.) This equation is obtained from the equation of continuity in exact form, and an approximate force equation suitable to study wave motion. The force equation is obtained when substituting for the pressure gradient an expression obtained by linearizing the diffusion terms due to viscosity, heat conduction and relaxation in the heat exchange equation and the equation of state, but keeping all nonlinear terms.

3. Interaction - Quasilinear Approximation

We now consider interaction between two waves with finite, but moderate amplitude, such that the quasilinear approximation can be used by substituting the linearized solution in the nonlinear source term. We then have

$$\nabla^2 P + \chi^2 P = Q \qquad (6)$$

as our basic equation in the generated pressure. Here Q is proportional to the product of two collinear axisymmetric soundbeams with wave numbers k_1, $k_2 (k_1 > k_2)$ and absorption coefficients α_1, α_2. Further $\chi = k + i\alpha$, $(k = k_1 - k_2)$ is the complex wave number of the generated difference frequency sound.

We substitute $P = e^{i\chi z} q$, and introduce the scaling $z = L\sigma$, $(x^2+y^2)^{\frac{1}{2}} = a\xi$, and obtain

$$\frac{1}{kL}\frac{\partial^2 q}{\partial \sigma^2} + 2i\,\frac{\chi}{k}\frac{\partial q}{\partial \sigma} + \frac{L}{ka^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi}\right) q = \frac{L}{k}\, e^{-i\chi z}\, Q \ . \qquad (7)$$

We seek expansions of q in powers of a small parameter, say η, which then depends on the various parameters determining the problem. Here more than one parameter is involved, and therefore several ordering possibilities exist. A principal requirement, however, is that terms accounting for the different effects of diffraction, absorption and nonlinearity are equally important to leading order in η. If we assume L/ka^2 finite, we may expand in powers of $\eta = 1/kL$ (when $L = L_A = (\alpha_1 + \alpha_2 - \alpha)^{-1}$, $L/ka^2 = N_A^{-2}$, where N_A is the aperture number introduced by Vestrheim[3] in his classification of parametric acoustic arrays). In the limit $\eta \to 0$, we obtain the following simplified equation

$$2i\,\frac{\chi}{k}\frac{\partial q}{\partial \sigma} + \frac{L}{ka^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi}\right) q = \frac{L}{k}\, e^{-i\chi z}\, Q \ . \qquad (8)$$

This can readily be solved by introducing the Hankel transform of zeroth order with respect to the lateral coordinate. The solution is matched to an inner solution which satisfies the boundary condition q = 0 at z = 0, and which is obtained by keeping the second derivative term in Eq. (7).

If we assume nonspreading carrier waves and approximate the source term Q with a Gaussian profile $\left(n \text{ constant}, \ \Lambda = \rho_o c_o^{-2}\left(d^2P/d\rho^2\right)_{\rho=\rho_o}\right)$,

$$Q = \frac{\Lambda+2}{2} \frac{k^2}{\rho_o c_o^2} p_o^2 \exp\left(-\sigma - n\xi^2\right) \quad , \tag{9}$$

we find the following solution

$$q(\sigma,\xi) = -i \frac{\Lambda+2}{8\pi} \frac{p_o^2}{\rho_o c_o^2} (ka)^2 e^{-\sigma} \int_o^\sigma \frac{\exp\left(u - \frac{\sigma^2\theta^2}{iu + N_A^2/2n}\right)}{iu + N_A^2/2n} du , \tag{10}$$

where $\theta = (kL_A/2)^{\frac{1}{2}} \xi/\sigma$, about the ratio of the observation angle to the 3 dB angle in the Westervelt model[4]. This is in accordance with results derived by Novikov et.al.[5] on the basis of Zabolotskaya and Khokhlov's equation, and used by Hobaek and Tjøtta[6] to calculate the variation of the 3 dB angle with range.

So far we have ignored the oscillations in amplitude and phase angle of the carrier beams. The model is therefore expected to be good only for $z \leq O(r_1/2\pi)$, or in practice for a parametric array with $L_A \leq r_1/2\pi$. Neglecting these oscillations may lead to significant errors if interaction in the transient region is an important contribution to the total generated sound. Equation (1) should be used to calculate Q in this region, but this leads to a more involved mathematical analysis. On the other hand, an indication of how this effect comes out is readily seen by modifying the Westervelt model: let us assume a line distribution of sources on the axis, with amplitude given by Eq. (2) instead of a constant. At observation distances r such that the Born approximation is valid, we have

$$q(r,\theta) = -i \frac{\Lambda+2}{4} \frac{p_o^2}{\rho_o c_o^2} (ka)^2 \frac{L_A}{rX}\left[1 + i\pi N_A\left(\frac{X}{2}\right)^{\frac{1}{2}} H_1^{(1)}\left(N_A(2X)^{\frac{1}{2}}\right)\right] \quad , \tag{11}$$

where θ is the observation angle and $X(\theta) = i + 2kL_A \sin^2\theta/2$. This is the scattering formula obtained by Westervelt with a correction factor given by the parenthesis. On the axis, this factor may be expressed in terms of Kelvin functions.

The correction is about 1 when $N_A \to \infty$, 2 when $N_A \to 0$, and 1.2 in amplitude for $N_A = 1.56$ (the value in Vestrheim and Hobaek's experiment[7]). This should indicate that the phase and amplitude oscillations in the carrier beams lead to a reduction in the amplitude of the generated difference frequency sound at large ranges, especially for high and moderate values of N_A, i.e., for array lengths which are short compared to the Rayleigh distance of the generated sound. For finite N_A the beamwidth is also reduced (12% for $N_A = 1.2$).

The case of spherically spreading carrier waves is discussed thoroughly elsewhere[8] and we here limit ourselves to only indicate some of the results for the directivity of the generated sounds (see Fig. 2). The paper referred to also contains the solution for a model with carrier beams formed by plane waves. Our results in Fig. 2 show good overall agreement with observations of the half power angle by various authors, taken at different ranges from the piston source. They also show that the directivity does not approach the product directivity of the two carrier beams as $N_D = (kL_A)^{\frac{1}{2}}/k_1a$ increases, within the parameter ranges used. Only if $(-\ell n r_1)k/k_1 \gg 1$ will the directivity of the generated sound approach the product directivity. The reason for this is, of course, the important property of the difference frequency sound wave, that it contains a "linear part" (i.e., solution of the homogeneous wave equation) which is propagated to the farfield region where it dominates the field.

4. Weak Nonlinearity - Expansion Based on Burgers' Equations

The most general case is studied by working with the Fourier transform of Eq. (4) and using matching of asymptotic expansion as previously indicated. Here, however, we limit ourselves to discuss the case where the effect of diffraction can be ignored, to leading order in the expansion parameter, in the nonlinear terms. Good models are then obtained by inserting the unidirectional solution of the Burgers' equation into these source terms.

The exact solution of Burgers' equation can, in the nontransient time region, be expanded in a double power series in the acoustic Reynolds numbers, $R_i = M_i/S_i = V_{oi}c_o/D\omega_i$ $(i = 1,2)$ of the two waves (Faltinsen and Tjøtta[9]). The series is convergent for all finite values of R_i. The three first terms in the expansion lead to the nonlinear taper functions, T_1 and T_2 for the fundamental harmonic components,

$$T_1 = 1 - \frac{\beta R_1^2}{2} e^{-2\alpha_1 z}\left[\sinh(\alpha_1 z)\right]^2 + \frac{\beta R_2^2}{2} e^{-2\alpha_2 z}\left[1 - \cosh\left(2\sqrt{\alpha_1\alpha_2}\,z\right)\right] \quad (12)$$

and similar expression for T_2. Here $\beta = 1 + \frac{1}{2}\rho_o c_o^{-2}\left(\frac{d^2P}{d\rho^2}\right)_{\rho=\rho_o}$
If $\alpha_2 \simeq \alpha_1$ and $R_2 \simeq R_1$ we get

$$T_1 \simeq 1 - \frac{3}{8}\left(\beta R_1\right)^2\left(1 - e^{-2\alpha_1 z}\right)^2 \quad (13)$$

which is the result derived by Hobaek and Vestrheim[10] from energy considerations and by applying the Manley-Rowe equations. But the correct source term in Eq. (4), or in the Burgers' equation, is not obtained by simply substituting $V_{oi}e^{-\alpha_i z}T_i(z)\cos\omega_i\tau$ $(i = 1,2,\ \tau = t - z/c_o$ retarded time) for the velocity fields of the carrier beams (with vanishing field outside the beams), which is a common practice. The reason for this is, of course, that the velocity field to this order also contains terms with other frequency combinations. Inserting the total field into the nonlinear source term, several of these additional terms will contribute to the difference frequency term to the same order of magnitude as the taper functions. Interaction between the fundamentals and third order terms of the form $\cos[(2\omega_1 - \omega_2)\tau]$ and $\cos[2\omega_2 - \omega_1)\tau]$, and of $\cos(2\omega_1\tau)$, $\cos(2\omega_2\tau)$ with $\cos[(\omega_1 + \omega_2)\tau]$ all produce important difference frequency terms.

It is straightforward to obtain the complete source term, on the basis of the third order solution given by Faltinsen and Tjøtta[9], but the expressions are in the general case algebraic involved, and will therefore not be presented here. By accounting for terms to this order, one can explain the finite amplitude effects observed by Hobaek and Vestrheim[10]. Qualitatively the effect leads in their case ($L_A < r_1/2\pi$) to stronger attenuation of the source term, thus reducing the effective array length. The result is asymptotically a broader beam for the generated sound.

Finally, we note that the one-wave taper function is obtained by putting $V_{o2} = 0$ in Eq. (12). For $\alpha_2 = \alpha_1$, and $R_2 = R_1$, the expression is similar to Eq. (13) with the factor 3/8 replaced by 1/8. This, and the discussion above, may indicate the errors introduced by inserting a one-wave solution with taper functions into the nonlinear source term, and thereby calculating the generated difference frequency sound. The Bartram model[11] is based on such substitutions, although with stronger nonlinearity than given by the first terms of our expressions.

5. Strong Nonlinearity - Asymptotic Solution of Burgers' Equation

Blackstock[12] has used Burgers' equation to derive the Fay-type solution for a single, finite amplitude wave. We have applied this approach to the case of two interacting waves[13] and obtain for $R_1 >> 1$ a simple asymptotic formula for the generated difference frequency sound in the farfield region (outside the interaction region).

The solution of the Burgers' equation can be written

$$V = \frac{2Dc_o^2}{\beta} \frac{\partial}{\partial \tau} \ln \phi \, , \qquad (14)$$

where

$$\phi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds\, e^{-s^2 + \beta R_1 \sin\left[\omega_1\left(\tau - 2\sqrt{Dz}\, s\right)\right] + \beta R_2 \sin\left[\omega_2\left(\tau - 2\sqrt{Dz}\, s\right)\right]} + T(t) \qquad (15)$$

Here T(t) denotes transient terms that are ignored in the following, as $T(t) \to 0$ for $t \to \infty$, z fixed . The linearized boundary condition $V = V_{o1}\cos\omega_1 t + V_{o2}\cos\omega_2 t$ for $t > 0$, $z = 0$ are imposed.

Solutions can now be obtained by expanding the integrand in Eq. (15) in a Fourier-Bessel series and integrating term by term. Substituting for the Bessel functions the asymptotic formula derived by Blackstock we obtain

$$V = \frac{2Dc_o^2\left(\omega_1 - \omega_2\right)}{\beta} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sinh n\Omega} \sin\left[n\left(\omega_1 - \omega_2\right)\tau\right] \qquad (16)$$

if $\omega_1 \simeq \omega_2$ and $z >> \alpha_1^{-1}$. Here

$$\Omega = \left(\omega_1-\omega_2\right)^2 Dz + \frac{1}{2\beta R_1} + \frac{1}{2\beta R_2} = \left(\alpha + \frac{\alpha_1}{\sigma_1} + \frac{\alpha_2}{\sigma_2}\right) z , \qquad (17)$$

where σ_1 and σ_2 are coordinates in unit of shock-formation distance. The amplitude of the difference frequency term is thus $-2Dc_o{}^2(\omega_1 - \omega_2)/\beta \sinh \Omega$.

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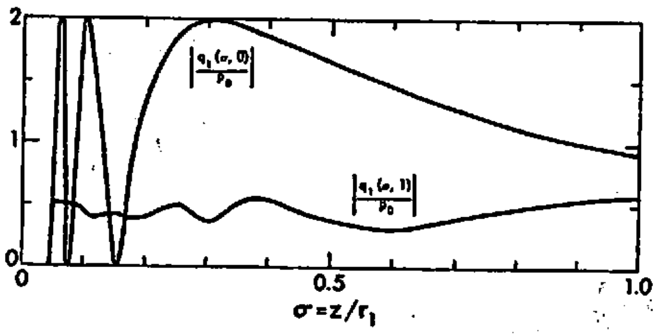

FIGURE 1

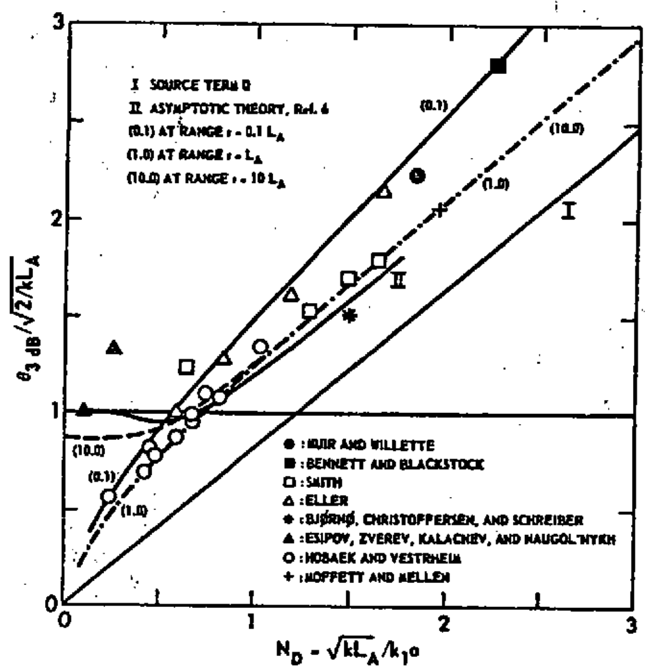

FIGURE 2