# SOME RESULTS ON A REVISITED BOUNCING BALL MODEL EXCITED BY A STATIONARY CONTINUOUS TIME CORRELATED STOCHASTIC PROCESS

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The popular bouncing ball model, which consists in a ball submitted to the gravitational field and bouncing vertically on a vibrating plate with inelastic impacts, is under study in this paper. Contrary of most of studies which assume a harmonic vertical motion of the plate, one considers random excitations of the ball induced by the plate motion. More precisely, we consider the dynamic behaviour of a revisited stochastic version of the bouncing ball model, by introducing the table displacement as a continuous time Gaussian random process with tunable correlation time. The memory effect of the excitation is then analysed, by investigating the dynamic behaviour through numerous numerical simulations. One shows that the dynamic behaviour is not only governed by the restitution coefficient at impacts and the dimensionless excitation amplitude level, but also by the correlation time of the excitation process. One highlights the relevant parameter that distinguishes the well-differentiated dynamic regimes. Quickly says, this dimensionless parameter depends for the essential from the bandwidth of the excitation signal.
Keywords: nonlinear dynamics, random process, memory

## 1.  Introduction

The popular bouncing ball (BB) model, which consists in a ball submitted to the gravitational field and bouncing vertically on a vibrating plate with inelastic impacts, has been widely studied in the last decades. This is due to both its simplicity and the amazing richness of its dynamics, from harmonic to chaotic, through subharmonic and quasi-periodic responses. It is now one of the paradigms for nonlinear dynamics and chaos (see, *e.g.* [1, 2] for BB in textbooks). Most of the studies achieved to date consider harmonic vertical motion of the plate. On the contrary, few of them include random vibrations of the plate, in spite of its undeniable relevance. Moreover, the excitation induced by the plate motion at successive bounces is generally assumed to be a discrete Markovian memoryless stochastic process. However, the real plate motions are always characterized by a finite auto-correlation time $t_{corr}$ below which the signal keeps memory of its previous values. The Markovian assumption of independent successive impacts corresponds to the fact that the ballistic flight time of the ball between two successive bounces is much larger than $t_{corr}$. This is the case with the so-called Chirikov conditions [3]. Conversely, for two bounces separated by a short flight time, the two relevant plate velocities can be strongly correlated. Thus, in regimes in which short flight times are dominant, the standard Markovian approach is expected to fail to capture the BB model dynamics.

In this context, the main purpose of this study is to characterize the BB model dynamics with stochastic excitation, when memory effects cannot be neglected.

## 2.    The dynamic model

Consider the popular BB model (see Figure 1) consisting on a point-like ball of finite mass bouncing vertically under the action of gravity, $g$, on an infinitely massive vibrating plate, the originality of our model is to introduce, for the vibrating plate, a correlated stochastic motion, $h(t)$, with tunable correlation time, $t_{corr}$. To this end, $h(t)$ is obtained from an uncorrelated Gaussian white noise $\psi(t)$, filtered by a second-order filter as

$$\ddot{h} + 2\zeta\Omega\dot{h} + \Omega^2 h = \psi(t) \tag{1}$$

with $\Omega$ being the center frequency of the filter and $\zeta$ its damping coefficient. Note that $\zeta$ is related to the frequency contents of the signal because the bandwidth of its power spectrum density (PSD), $S_{hh}(\omega)$ is equal to $2\zeta\Omega$. The autocorrelation function $< h(t)h(t + \tau) >$ of $h$ is equal to $\sigma_h^2 \exp(-\zeta\Omega|\tau|) f(\tau)$ with $f$ a periodic function and $\sigma_h$ the standard deviation of $h$, so the correlation time can be defined as $t_{corr} = 1/\zeta\Omega$. To avoid infinite energy in the acceleration signal the displacement is further filtered by a first-order low-pass filter with a cutoff frequency higher than $\Omega$. Typical simulated PSDs are shown in Figure 2 for the two cases, narrow and broadband.
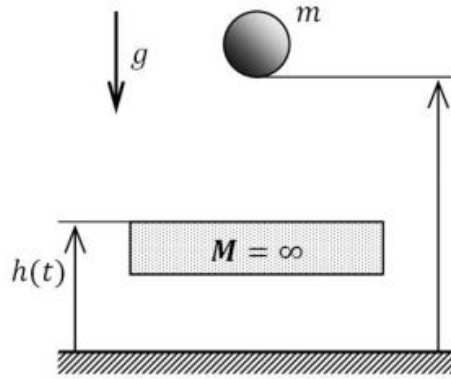


Figure 1. Sketch of the bouncing ball model.

Now, for any generated excitation signal, we then solve the bouncing ball problem by calculating the values of the post impact velocity, $v_n$, and instant, $t_n$, of the all-successive impacts. In practice, we solve the following equations:

$$t_{n+1} = t_n + \theta_n \tag{2}$$

$$v_{n+1} = -e(v_n - g\theta_n) + (1 + e)w_{n+1} \tag{3}$$

with $e$ the restitution coefficient introduced to take into account the partially inelastic impact characteristic, $w_n$ the plate velocity at impact, and $\theta_n$ the flight time obtained from the following equation:

$$-\frac{1}{2}g\theta_n^2 + v_n\theta_n + h_n - h_{n+1} = 0 \tag{4}$$

In fact, Equations (2) and (3) define the classical Poincaré map for the BB model. A dimensional analysis shows that the system is governed only by three dimensionless parameters, i.e. the restitution coefficient $e$, the reduced plate acceleration defined by $\Lambda = \sigma_w^2/g\sigma_h$ and the dimensionless correlation time $\tau_{corr} = \Omega t_{corr}$. On this basis, we have performed simulations for a large number of values of this triplet [4].
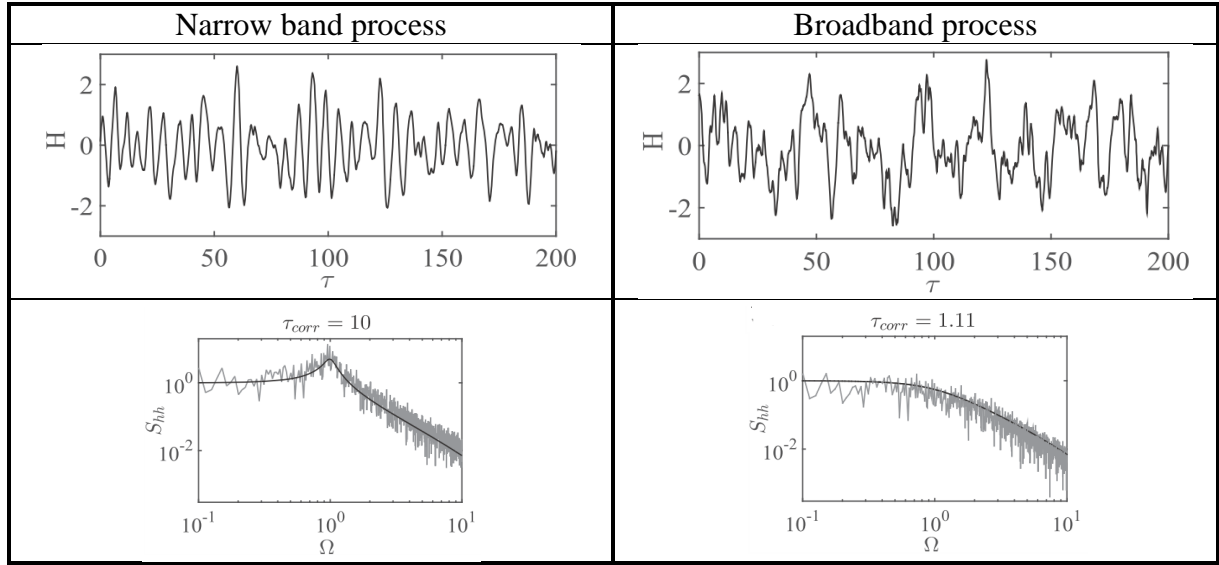
Figure 2. Typical realizations of the dimensionless plate time-displacement and its power spectral densities (PSD): narrow and broadband cases.

## 3. Results

Typical probability density functions (pdf) of the dimensionless take-off velocity $V_n = v_n/\sigma_w$ are shown in Figure 3. Wood and Byrne have studied the case of a completely uncorrelated Markovian excitation [5], and their results (velocity quoted $V_{WB}$) are used as a reference to highlight the differences brought by our improved model which includes the correlation in the excitation. As we can see, memory effects become negligible when individual flight times are larger than the correlation time. This case is favoured for large excitations $\Lambda$ and/or short correlation time $\tau_{corr}$.
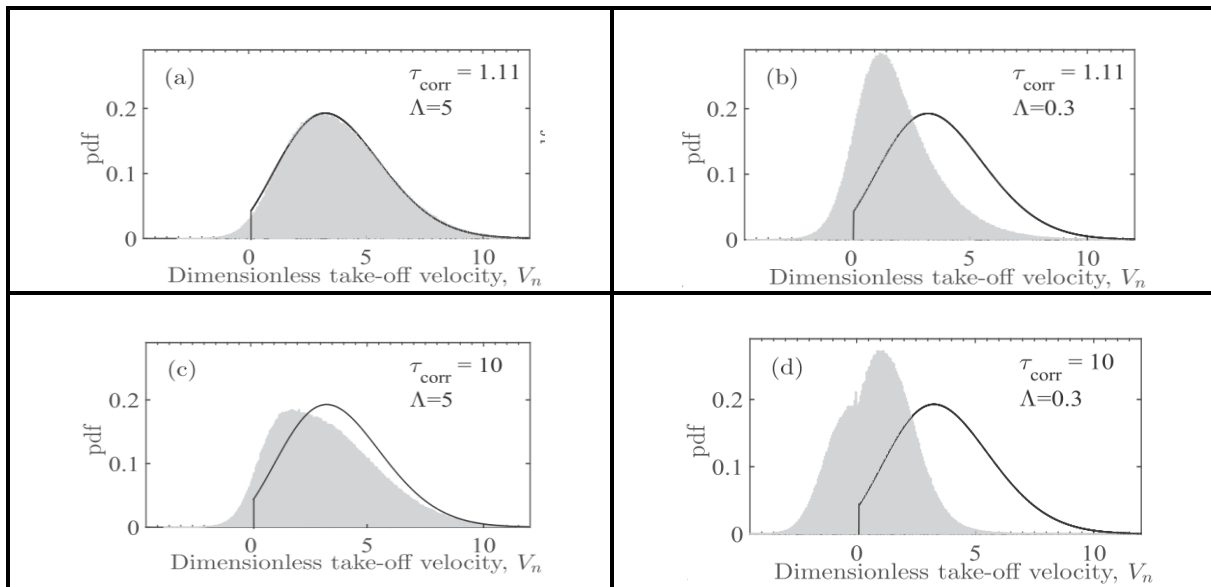


Figure 3. Typical pdf of the dimensionless take-off velocity for selected $\Lambda$ and $\tau_{corr}$, with the example of $e = 0.8$. Solid lines correspond to the Wood and Byrne results.

A detailed analysis [6] shows that the relevant parameter consists on the ratio of the Markovian mean flight time of the ball and the mean time between successive peaks in the plate motion. This dimensionless parameter depends on the bandwidth of the excitation signal. When the parameter is large, the Markovian approach is appropriate; but for low levels, memory effects become not negligible leading to a significant decrease of jump durations; and finally at smallest values of the ratio, chattering occurs.

## 4.    Acknowledgment

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