MAXIMUM LIKELIHOOD ESTIMATION OF TIME-VARYING DELAY

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I. INTRODUCTION

This paper outlines the theoretical solution to the problem of maximum likelihood (ML) estimation of time-varying delay d(t) between a random signal s(t) received at one point in the presence of uncorrelated noise and the time-delayed, scaled version $\tilde{a}s(t - d(t))$ of that signal received at another point in the presence of uncorrelated noise. The signal is assumed to be a sample function of a nonstationary Gaussian random process and the observation interval is arbitrary. The analysis of this paper represents a generalization of that of Knapp and Carter [1], who derived the ML estimator for the case that the delay is constant, $d(t) = d_0$, the signal process is stationary, and the received processes are observed over the infinite interval $(-\infty, +\infty)$. A more detailed presentation of the topic of this paper appears in [2].

We model the problem of time-varying delay estimation as follows:

A vector of real waveforms

$$\underline{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} s(t) \\ \tilde{a}s(t - d(t)) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \qquad (1.2)$$

is observed on the interval $[T_i, T_f]$. For convenience, we define $\underline{r}(t)$ as zero for t outside this interval. The signal s(t) is a sample function of a zero-mean Gaussian random process having covariance function

$$R_s(t_1,t_2) = E\left\{s(t_1)s(t_2)\right\}. \qquad (1.3)$$

The delayed and attenuated signal $\tilde{a}s(t - d(t))$ is related to s(t) through a non-random but unknown invertible linear operator

$$\mathcal{L}_{d(t),\tilde{a}}\left\{s(t)\right\} = \tilde{a}s(t - d(t)) \qquad (1.4)$$

The noise waveforms $w_1(t)$ and $w_2(t)$ are sample functions of white Gaussian random processes having covariance functions

$$R_{w_1}(t_1,t_2) = R_{w_2}(t_1,t_2) = \frac{N_o}{2} \delta(t_1 - t_2) . \quad (1.5)$$

The signal process and noise processes are mutually independent. The attenuation factor $\tilde{a}$ and delay function d(t) appearing in (1.2) and (1.4) are nonrandom but unknown. Since d(t) represents delay, we will assume throughout this paper that $d(t) \geq 0$. The attenuation constant $\tilde{a}$ can be any nonzero real number. The problem is to estimate d(t) and $\tilde{a}$.

II. THE LOG LIKELIHOOD FUNCTION

The first step in the derivation is to represent d(t) as a parameter vector $\underline{d} = (d_1, d_2, \ldots)$ by expanding it into a series using any convenient basis $\{\psi_i(t)\}$. We can then write

$$s(t;\underline{d}) \triangleq s(t - \sum_{i=1}^{\infty} d_i \psi_i(t)) . \quad (2.1)$$

It follows from notation (2.1) that

$$s(t;\underline{0}) = s(t) . \quad (2.2)$$

We now write $\underline{r}(t)$ of (1.2) as

$$\underline{r}(t) = \underline{s}(t;\underline{d},\tilde{a}) + \underline{w}(t) \quad (2.3)$$

where

$$\underline{r}(t) = (r_1(t)\ r_2(t))^T , \quad (2.4a)$$

$$\underline{s}(t;\underline{d},\tilde{a}) \triangleq (s(t;\underline{0})\ \tilde{a}s(t;\underline{d}))^T , \quad (2.4b)$$

and

$$\underline{w}(t) = (w_1(t)\ w_2(t))^T \ . \tag{2.4c}$$

It follows for $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, $\underline{r}(t)$ of (2.4) is a Gaussian random vector process having mean zero and 2 x 2 matrix covariance function

$$K_{\underline{r};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A}) \triangleq E\left\{\underline{r}(t)\underline{r}^T(u)\,\middle|\,\underline{d} = \underline{D},\tilde{a} = \tilde{A}\right\}$$

$$= E\left\{\underline{s}(t;\underline{D},\tilde{A})\underline{s}^T(u;\underline{D},\tilde{A})\right\} + E\left\{\underline{w}(t)\underline{w}^T(u)\right\}$$

$$= K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A}) + \frac{N_o}{2} I\ \delta(t-u) \tag{2.5}$$

where I is the 2 x 2 identity matrix.

We proceed by representing vector process $\underline{r}(t)$ as an infinite dimensional vector $\underline{r}$ using the generalized Karhunen-Loeve expansion [3, pp. 221-223]. As shown in [2], this leads to the log-likelihood function

$$\ln\Lambda(\underline{D},\tilde{A}) = \ell_R(\underline{D},\tilde{A}) + \ell_B(\underline{D},\tilde{A}) \tag{2.6}$$

where

$$\ell_R(\underline{D},\tilde{A}) = \frac{1}{N_o} \int_{T_i}^{T_f}\int_{T_i}^{T_f} \underline{r}^T(t)\underline{H}_n(t,v;\underline{D},\tilde{A})\underline{r}(v)dtdv \tag{2.7}$$

and

$$\ell_B(\underline{D},\tilde{A}) = -\frac{1}{2}\int_{T_i}^{T_f} \mathrm{Tr}[\underline{H}_c(t,t;\underline{D},\tilde{A})]dt \ . \tag{2.8}$$

The 2 x 2 matrix function $\underline{H}_n(t,v;\underline{D},\tilde{A})$ in (2.7) is the solution to

$$K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A}) - \frac{N_o}{2}\underline{H}_n(t,u;\underline{D},\tilde{A}) - \int_{T_i}^{T_f} \underline{H}_n(t,v;\underline{D},\tilde{A})K_{\underline{s};\underline{d},\tilde{a}}(v,u;\underline{D},\tilde{A})dv$$

$$= \underline{0}\ ; \quad T_i \le t_i,\ u \le T_f\ . \tag{2.9}$$

In order to interpret $\underline{H}_n(t,v;\underline{D},\tilde{A})$, define

$$\hat{\underline{s}}_n(t;\underline{D},\tilde{A}) = \int_{T_i}^{T_f} \underline{H}_n(t,v;\underline{D},\tilde{A})\underline{r}(v)dv,\ T_i \le t \le T_f\ . \tag{2.10}$$

It can be shown [2] that $\hat{\underline{s}}_n(t;\underline{D},\tilde{A})$ is the LMMSE noncausal estimate of $\underline{s}(t;\underline{D},\tilde{A})$ when $\underline{D}$ and $\tilde{A}$ are the true values of $\underline{d}$ and $\tilde{a}$ respectively. Similarly, the 2 x 2 matrix function $\underline{H}_c(t;v;\underline{D},\tilde{A})$ in (2.8) is the impulse response of the causal LMMSE estimate $\hat{\underline{s}}_c(t;\underline{D},\tilde{A})$ of $\underline{s}(t;\underline{D},\tilde{A})$ given that $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$. $\underline{H}_c(t,v;\underline{D},\tilde{A})$ and $\hat{\underline{s}}_c(t;\underline{D},\tilde{A})$ are equal to $\underline{H}_n(t,v;\underline{D},\tilde{A})$ and $\hat{\underline{s}}_n(t;\underline{D},\tilde{A})$ respectively when $T_f = t$.

The values of $\underline{D}$ and $\tilde{A}$ jointly maximizing the log-likelihood function (2.6) are by definition the maximum likelihood estimates $\hat{\underline{D}}$ and $\hat{A}$ of $\underline{d}$ and $\tilde{a}$ respectively from $\underline{r}(t)$, $T_i \le t \le T_f$. The maximum likelihood estimate $\hat{D}(t)$ of time-varying delay $d(t)$ is the waveform represented by $\hat{\underline{D}}$.

III. THE MATRIX IMPULSE RESPONSE $\underline{H}_n(t,v;\underline{D},\tilde{A})$

In this section we derive a simple explicit form for the matrix impulse response $\underline{H}_n(t,v;\underline{D},\tilde{A})$. It is relatively difficult to obtain this form by solving equation (2.9). The constructive approach taken here has the advantage of being both mathematically and conceptually simple.

The first step in the derivation of $\underline{H}_n(t,v;\underline{D},\tilde{A})$ is to (noncausally) transform $\underline{r}(t)$, $T_i \leq t \leq T_f$, into the vector process $\underline{r}'(u)$, $f(T_i) \leq u \leq T_f$, where

$$\underline{r}'(u) = \begin{bmatrix} \tilde{A}^{-1} r_2(\beta(u)) \\ 0 \end{bmatrix}; \; f(T_i) \leq u \leq T_i \qquad (3.1a)$$

$$\underline{r}'(u) = \frac{1}{2} \begin{bmatrix} r_1(u) + \tilde{A}^{-1} r_2(\beta(u)) \\ r_1(u) - \tilde{A}^{-1} r_2(\beta(u)) \end{bmatrix}; \; T_i \leq u \leq f(T_f) \qquad (3.1b)$$

$$\underline{r}'(u) = \begin{bmatrix} r_1(u) \\ 0 \end{bmatrix}; \; f(T_f) \leq u \leq T_f \qquad (3.1c)$$

In the above,

$$f(t) = t - D(t) \qquad (3.2)$$

and $\beta(t)$ is the inverse of $f(t)$

$$\beta[f(t)] = t \; . \qquad (3.3)$$

In (3.1), $\tilde{A}$ can be regarded as an assumed value for the unknown relative attenuation constant $\tilde{a}$, and $D(t)$ as an assumed function for the unknown delay function $d(t)$. We naturally require $D(t) \geq 0$. The transformation $\underline{r}(t) \rightarrow \underline{r}'(u)$ is illustrated in Figure 3.1, where for simplicity in interpretation, the noise processes $w_1(t)$ and $w_2(t)$ have been drawn as small ripples.

An examination of equation (3.1) and Figure 3.1 will reveal that the transformation from $\underline{r}(t)$ to $\underline{r}'(u)$ is linear and invertible. Thus, $\underline{r}(t)$, $T_i \leq t \leq T_f$ can be recovered from $\underline{r}'(u)$, $f(T_i) \leq u \leq T_f$, using a linear transformation. It follows from the reversibility theorem [3, pg. 289] that the noncausal LMMSE estimate $\hat{\underline{s}}_n(t;\underline{D},\tilde{A})$ of equation (2.10) given $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, can be obtained from $\underline{r}'(u)$. Before describing the structure of the LMMSE estimator, it will be helpful to observe that if $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then, from equations (2.3), (2.4), and (3.1):

$$\underline{r}'(u) = \underline{0} \; ; \; u < f(T_i) \tag{3.4a}$$

$$\underline{r}'(u) = \begin{bmatrix} s(u) \\ 0 \end{bmatrix} + \begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} \; ; \; f(T_i) \le u \le T_f \tag{3.4b}$$

$$\underline{r}'(u) = \underline{0} \; ; \; T_f < u \, , \tag{3.4c}$$

where

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1} w_2(\beta(u)) \\ 0 \end{bmatrix} ; \; f(T_i) \le u \le T_i , \tag{3.5a}$$

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} w_1(u) + \tilde{A}^{-1} w_2(\beta(u)) \\ w_1(u) - \tilde{A}^{-1} w_2(\beta(u)) \end{bmatrix} ; \; T_i < u \le f(T_f) \, , \tag{3.5b}$$

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \begin{bmatrix} w_1(u) \\ 0 \end{bmatrix} ; \; f(T_f) < u \le T_f \; . \tag{3.5c}$$

It can be shown that the noncausal point LMMSE estimator of $s(t)$ from $\underline{r}'(u)$, $f(T_i) \le t$, $u \le T_f$, conditioned on $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, is given by the system in Figure 3.2, where $f(t,u;\underline{D},\tilde{A})$ is the impulse response of the noncausal point LMMSE estimator $\hat{n}_1(t)$ of $n_1(t)$ from $n_2(u)$.

$$\hat{n}_1(t) = \int_{f(T_i)}^{T_f} f(t,u;\underline{D},\tilde{A}) n_2(u) du \; ; \; f(T_i) \le t \le T_f \tag{3.6}$$

and $g_n(t,u;\underline{D},\tilde{A})$ is the impulse response of the noncausal point LMMSE estimator $\hat{s}_n(t)$ of $s(t)$ from $s(u) + n_1(u) - \hat{n}_1(u)$,

$$\hat{s}_n(t) = \int_{f(T_i)}^{T_f} g_n(t,u;\underline{D},\tilde{A})\,[s(u) + n_1(u) - \hat{n}_1(u)]du \ ; \ f(T_i) \le t \le T_f \ . \quad (3.7)$$

A proof of this assertion is given in [2].

The LMMSE estimator of $\tilde{a}s(t - d(t))$, $T_i \le t \le T_f$, from $\underline{r}(u)$, $T_i \le u \le T_f$, conditioned on $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, follows easily from the fact that $\tilde{a}s(t - d(t))$ is a linear transformation of $s(t)$. Because all available data have been used to obtain $\hat{s}_n(t)$, $f(T_i) \le t \le T_f$, the noncausal LMMSE estimate of $\tilde{a}s(t - d(t))$, given $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, is simply the scaled and delayed version of $\hat{s}_n(t)$ of (3.7), namely, $\tilde{A}\hat{s}_n(t - D(t))$.

The specific form of the impulse response $f(t,u;\underline{D},\tilde{A})$ turns out to be [2]

$$f(t,u;\underline{D},A) = k(u)\ \delta(t - u) \quad (3.8)$$

where

$$k(u) \triangleq \begin{cases} \dfrac{\tilde{A}^2 - [1 - \dot{D}(\beta(u))]}{\tilde{A}^2 + [1 - \dot{D}(\beta(u))]} \ ; & T_i < u \le f(T_f) \\ 0 \ ; \ \text{otherwise.} & \end{cases} \quad (3.9)$$

Consequently

$$\hat{n}_1(t) = k(t)n_2(t) \ . \quad (3.10)$$

An equation specifying $g_n(t,u;\underline{D},\tilde{A})$ can be obtained by using the fact that $g_n(t,u;\underline{D},\tilde{A})$ is the LMMSE estimator of $s(t)$, $f(T_i) \le t \le T_f$, from

$$z(u) = s(u) + n(u) \ ; \ f(T_i) \le u \le T_f \ . \quad (3.11)$$

where

$$n(u) \triangleq n_1(u) - \hat{n}_1(u) \ ; \ f(T_i) \le u \le T_f \ . \qquad (3.12)$$

The noise process n(u) is zero mean, and uncorrelated with the signal process s(u). Its covariance function is [2]

$$E\left\{n(t)n(u)\right\} = Q(u)\delta(t-u) \ ; \ f(T_i) \le u \le T_f \qquad (3.13)$$

where

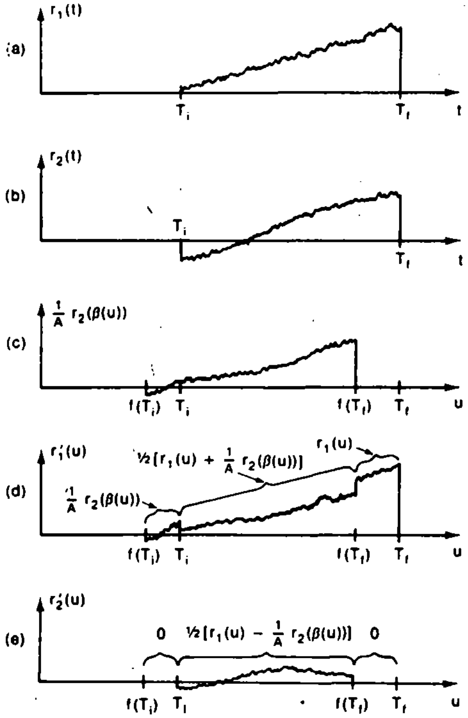
$$Q(u) = \begin{cases} \frac{N_o}{2}\,[1 - \dot{D}(\beta(u))] \ ; \ f(T_i) \le u \le T_i \\ \frac{N_o\,[1-\dot{D}(\beta(u))]}{2\left[\tilde{A}^2 + [1-\dot{D}(\beta(u))]\right]} \ ; \ T_i < u \le f(T_f) \\ \frac{N_o}{2} \ ; \ f(T_f) < u \le T_f \ . \end{cases} \qquad (3.14)$$

This leads directly to the equation

$$R_s(t,u) = \int_{f(T_i)}^{T_f} g_n(t,\sigma;\underline{D},\tilde{A})\, R_s(\sigma,u)\, d\sigma$$

$$+ Q(u)g_n(t,u;\underline{D},\tilde{A}) \ ; \ f(T_i)<t,u < T_f \ . \qquad (3.15)$$

We have now specified the structure of $\underline{H}_n(t,u;\underline{D},\tilde{A})$. This structure is shown in Figure 3-3, where the filter $g_n(t,u;\underline{D},\tilde{A})$ is the solution to (3.15) and where k(t) is given by (3.9). By tracing through this structure, we can derive the explicit form for the individual entries $h_{ij}(t,u;\underline{D},\tilde{A})$ in $\underline{H}_n(t,u;\underline{D},\tilde{A})$. Upon setting $T_f = t$ we then obtain the explicit form for the individual entries in $\underline{H}_c(t,u;\underline{D},\tilde{A})$. This leads to the explicit solution for $\ell_B(\underline{D},\tilde{A})$ via (2.8).



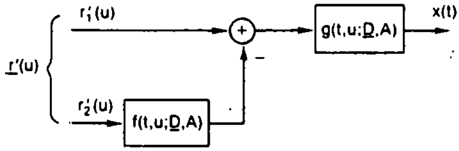


Fig. 3.2 Estimator of s(t). When $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then x(t) is the noncausal LMMSE estimator of s(t) from $\underline{r}'(u)$.

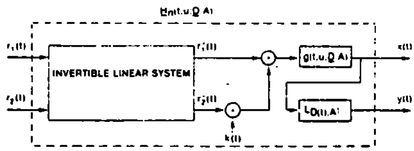


Fig. 3.1 Invertible Linear Transformation

Fig. 3.3 System $\underline{H}_n(t,u;\underline{D},\tilde{A})$

REFERENCES

[1] Charles H. Knapp and G. Clifford Carter, "The Generalized Correlation Method for Estimation of Time Delay," IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP-24, No. 4, August 1976, pp. 320-327.

[2] John A. Stuller, "Maximum-Likelihood Estimation of Time-Varying Delay," IEEE Transactions on Acoustics, Speech, and Signal Processing, In Press.

[3] Harry L. Van Trees, "Detection, Estimation and Modulation Theory, Part I," John Wiley and Sons, Inc., New York, London Sidney, 1968.