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A SYSTEMATIC APPROACH TOWARD THE SOLUTION OF UNDERWATER SOUND PROPAGATION PROBLEMS

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ABSTRACT A treatment is given to the mathematical problem depicting wave propagation through layered media. The solution procedure of the resulting integral equation is iterative. Unlike other iterative schemes, the present iterations are guaranteed to converge to the unique solution. The numerical calculations on the computer are recursive in character and are thus compatible with machine operations. This is an attractive feature in a real time system.

1. INTRODUCTION

In many models of the ocean environment, the medium is presumed to vary principally along an axis normal to the boundaries. The relations between such medium properties and the propagating wave within are not generally available in terms of known functions. Approximate methods exist but have constraints on the region of their validity. A direct numerical solution is helpful but it is not useful when scaling laws are to be applied or a general relationship is needed. The present work attempts to develop a complete solution which is generated by a series of successive approximations that are generally valid and that do lend themselves to easy implementation on the computer [1, 2 and their references].

2. SOLUTION TO A CANONICAL PROBLEM

The solution to a canonical problem in wave propagation is presented first, then applications are carried out to study cases that appear to be at variance with the basic procedure. The canonical problem consists of a plane wave $Ae^{-i(k_0x-wt)}$ incident on an arbitrary and continuously stratified region with planar boundaries [Figure 1]. The explicit composition of the reflected, transmitted and propagated waves are derived. The solution is systematic and allows for (i) discontinuities in the acoustic properties at boundaries and arbitrary variation within, (ii) attenuation, (iii) all angles of incidence. The auxiliary mathematical constants are plainly amenable to physical interpretation.

When the traditional formulation is followed, a Fredholm integral equation results. Its Neumann series solution is valid when the inhomogeneous layer is thin and/or the perturbation is small. Though the Fredholm series solution is generally valid, it is encumbered by multiple integrations over large order determinants [4]. The present procedure avoids the preceding difficulties and remains applicable to thick layers which large perturbations. The present formulation yields a Volterra integral equation

$$u(x) = h(x)+H(a)q(x)+\lambda \int_0^x K(x,x')u(x')dx', \qquad (1)$$

where

$$h(x) = \frac{2A}{D}\left\{\left[u_2(a)+\frac{u_2'(a)}{ik_1}\right]u_1(x)-\left[u_1(a)+\frac{u_1'(a)}{ik_1}\right]u_2(x)\right\},$$

$$q(x) = \frac{1}{D}\left\{\left[u_1(0)-\frac{u_1'(0)}{ik_o}\right] u_2(x) - \left[u_2(0)-\frac{u_2'(0)}{ik_o}\right] u_1(x)\right\}.$$

$$D = \left[u_1(0)-\frac{u_1'(0)}{ik_o}\right]\left[u_2(a)+\frac{u_2'(a)}{ik_1}\right] - \left[u_2(0)-\frac{u_2'(0)}{ik_o}\right]\left[u_1(a)+\frac{u_1'(a)}{ik_1}\right]. \qquad (2)$$

$$H(a) = \lambda \int_0^a M(a,x')u(x')dx', \quad M(a,x') = -\left[K(a,x')+\frac{K'(a,x')}{ik_1}\right]$$

$$K(x,x') = \left[\frac{u_1(x)u_2(x')-u_2(x)u_1(x')}{u_1'(x')u_2(x')-u_2'(x')u_1(x')}\right] n(x'), \quad K'(a,x') = \left[\frac{u_1'(a)u_2(x')-u_2'(a)u_1(x')}{u_1'(x')u_2(x')-u_2'(x')u_1(x')}\right] n(x').$$

The term $\lambda n(x)$ denotes the perturbation in the index of refraction that renders the initial differential equation without a recognizable solution. The functions $u_1(x)$ and $u_2(x)$ are the recognizable solutions to a chosen part, $K_o(x)$, of the total index of refraction. The prime symbol on a function denotes a derivative with respect to x.

The iterative solution for the field u(x) is convergent with

$$u(x) = \sum_{n=1}^{N} u^{(n)}(x), \qquad (3)$$

$$u^{(n)}(x) = \lambda^{n-1} \int_0^x dx' K(x,x') \ldots \int_0^{x^{(n-2)}} K(x^{(n-2)},x^{(n-1)})\left[h(x^{(n-1)})+H(a)q(x^{(n-1)})\right]dx^{(n-1)},$$

and

$$H(a) = \frac{\int_0^a dx' M(a,x') \sum_{n=1}^{N} \lambda^n \int_0^{x'} dy' K(x',y') \ldots \int_0^{y^{(n-2)}} K(y^{(n-2)}),y^{(n-1)})h(y^{(n-1)})dy^{(n-1)}}{\left[1-\int_0^a dx' M(a,x') \sum_{n=1}^{N} \lambda^n \int_0^{x'} dy' K(x',y') \ldots \int_0^{y^{(n-2)}} K(y^{(n-2)},y^{(n-1)})q(y^{(n-1)})dy^{(n-1)}\right]} \qquad (4)$$

The size of N is controlled by the desired solution accuracy. The reflection and transmission coefficients are respectively

$$u_r = \frac{2A}{D}\left\{u_1(0)\left[u_2(a)+\frac{u_2'(a)}{ik_1}\right]-u_2(0)\left[u_1(a)+\frac{u_1'(a)}{ik_1}\right]-\frac{D}{2}\right\}+\frac{H(a)}{ik_oD}\left[u_1(0)u_2'(0)-u_2(0)u_1'(0)\right],$$

$$u_t = \frac{2Ae^{ik_1a}}{ik_1D}\left[u_1(a)u_2'(a)-u_1'(a)u_2'(a)\right]+\frac{H(a)e^{ik_1a}}{D}\left\{u_2(a)\left[u_1(0)-\frac{u_1'(0)}{ik_o}\right]\right. \qquad (5)$$

$$\left.-u_1(a)\left[u_2(0)-\frac{u_2'(0)}{ik_o}\right]\right\}+\lambda e^{ik_1a}\int_0^a K(a,x')u(x')dx',$$

The characteristic equation that governs the natural frequencies of the system is derived from equation (4) by nulling the source function. This yields

$$\left[1-\int_0^a dx' M(a,x')\sum_{n=1}^{N}\lambda^n\int_0^{x'} dy' K(x',y')\ldots\int_0^{y^{(n-2)}} K(y^{(n-2)},y^{(n-1)})q(y^{(n-1)})dy^{(n-1)}\right]=0 \quad (6)$$

The discrete normal modes corresponding to each natural frequency is generated by the same iterative form given in equation (3).

3. APPLICATIONS

The expanded applications treat formally a series of problems using the unified approach.

3.1 A Slowly Varying Profile

In many cases, an arbitrary form for $k_o{}^2K_o(x)$ may be taken where its variation is small over a wavelength. Then, the W-K-B approximation holds. Whenever the background profile changes sharply, such a solution is in error. For non-normal incidence also, there exist angles of incidence where $K_o(x)$ goes to zero and the W-K-B approximation fails [3].

To correct the induced errors, the problem is formulated so that the simplicity of the W-K-B approximation is retained as a first approximation. Improvement on that solution is obtained by treating the error as a perturbation. The resulting Volterra integral equation is:

$$u(x)=e^{(2\pi i/3)}L^{-1/2}(x)X^{1/2}(x)H^{(1)}_{1/3}(X)+u_r\left\{e^{(\pi i/6)}L^{-1/2}(x)X^{1/2}\left[H^{(1)}_{1/3}(x)+2e^{-(\pi i/3)}H^{(2)}_{1/3}(X)\right]\right\}$$

$$+\lambda\int_{-\infty}^{X}K(x,x')u(x')dx'$$

where

$$u_r=\frac{\lambda\pi}{8}e^{(2\pi i/3)}\int_{-\infty}^{\infty}L^{-1/2}(x')X^{1/2}(x')H^{(1)}_{1/3}\left[X(x')\right]n(x')u(x')dx',$$

$$K(x,x')=(\pi/8)\left[u_1(x)u_2(x')-u_1(x')u_2(x)\right]n(x').$$

The basic functions

$$u_1(x)=\left\{e^{(2\pi i/3)}/L^{1/2}(x)\right\}X^{1/2}(x)H^{(1)}_{1/3}\left[X(x)\right],$$

$$u_2(x)=\left\{e^{(\pi i/6)}/L^{1/2}(x)\right\}X^{1/2}(x)\left\{H^{(1)}_{1/3}\left[X(x)\right]+2e^{-(\pi i/3)}H^{(2)}_{1/3}\left[X(x)\right]\right\},$$

with $X(x)=\int_{X_0}^{X}L^{1/2}(x')dx'$, satisfy the equation for the background profile near the turning point and at infinity for a background profile $K_o(x)=[L(x)-\lambda n(x)]$, where $n(x)=[d^2r/dx^2]/r$ and $r(x)=X^{1/6}(x)/L^{1/4}(x)$, [3,4].

3.2 Point Source in a Layer

The problem of a point source radiating in a layer has been separated into two canonical problems [3] of the type defined here. The integral expressions for the field call in their integrands for specification of the propagated wave and the reflection coefficient, when a plane wave is incident on the inhomogeneous half spaces to either side of the source. The unknown terms in the integrands may be specified using equations (3-4).

3.3 Nonlinear Wave and Bifurcation Equation

A large amplitude wave produces a nonlinear behavior in the medium which in turn influences the propagation characteristics of that wave. The perturbation $n[x,u(x)]$ contains the nonlinearity as it is now functionally dependent on the field. The solution for $u(x)$ is given by

$$u^{(n)}(x)=\lambda^{n-1}\int_0^x dx'K\left\{x,x',n\left[x',u^{(n-1)}(x')\right]\right\}$$

and the bifurcation equation is

$$\alpha=\frac{\lambda}{W}\int_0^a u_2(x')n\left[x',\sum_{i=1}^{N}u^{(i)}(x',\alpha)\right]dx'$$

where W is the Wronskian.

3.4 Machine Implementation

For machine implementation, the preceding approach is dissected to reduce it to a recurrence relation. With the present approach, the solution is determined in about $n^2/2$ operations where n is the matrix size, as compared to $n^3/3$ operations used in Gauss elimination or n^2 operations for each iteration in a Gauss-Seidel solution. For unforced systems, the present process is efficient and requires $n^2/2$ operations in contrast to the n^3 operations required by techniques such as Leverrier-Faddeev method. Extended computer simulations show a reduction in solution time roughly proportional to n when compared to Gauss elimination or Leverrier-Faddeev and twice the iteration number when compared to a converging Gauss-Seidel result. In addition, the storage requirement is reduced by nearly one half and the numerical accuracy is increased by a factor of two. Results on extended computer simulations are presented to substantiate the accured improvement

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