An Adaptive Array with tailored robustness.

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1. Introduction

The design of robust optimal arrays has been pursued for many years. Gilbert and Morgan[1] maximised the directivity of an antenna with a constraint on the white noise response achieved by augmenting the diagonal of the covariance matrix of the isotropic noise. Lo et at[2] suggested the use of a bound on the array Q-factor or ratio of reactive to real input power in a transmitting antenna and this concept was applied to a sonar receiving array by Winkler and Schwartz[3]. Cox has discussed both of these constraint systems[4].

In the system to be described robutstness is achieved by the use of an inequality constraint on the weight vector W of the type $W^TAW \leq 1$ where A is a preset Hermitian matrix whose parameters will be determined from a-priori assumptions about the nature of likely errors in wanted signals. For the special case when A is an identity matrix the constraint system is very similar to that used by Gilbert and Morgan in its final result though the method of achieving the constraint is by a direct operation on the weight vector rather than by modification of the signals. It is therefore more suitable for on-line adaption rather than off-line optimisation since in the former case nothing may be known a-priori about the signal field.

The optimisation criterion to be used is minimisation of the total output power of the receiving array subject to a linear constraint on the weight vector of the form $C^TW = 1$ where C is a vector of complex phasors which define the look direction of the array, a system described by Frost[5] and others. The non-linear inequality $W^TAW \leq 1$ will be applied if necessary. Thus if R is an estimate of the covariance matrix of the signal+noise field:

$$R = \frac{1}{K} \sum_{1}^{K} X_k X_k^T \tag{1.1}$$

where X_k is the vector of complex sensor samples at time t_k then the processor simply solves the non-linear programming problem

$$\begin{aligned} &\text{Minimise output power:} && W^TRW \\ &\text{subject to} && C^TW = 1 \\ & && W^TAW \leq 1 \end{aligned} \tag{1.2}$$

Because of the convexity of quadratic forms it can be shown that these equations have a unique solution* $W=W_o$ corresponding to output power P_o and this power is our estimate of the signal strength in the direction corresponding to steering vector C. A variety of methods are avaiable writing such an algorithm[7].

In order to choose the matrix A the following observations are used:

(a) it is obviously necessary for the adaptive processor to have the requisite robustness to signal perturbations when only the wanted signal is present and there is no noise

(b) if the processor is designed for single signals it may be sufficiently robust when multiple signals are present coming from different directions. The single signal design covered in this paper is in fact quite straightforward and the non-linear constraint system evolved deals quite successfully with multiple signals as shown by simulations given only that different arrivals are uncorrelated. It will not handle correlated arrivals as might be caused, for example, by multipath propagation.

2. Single signal analysis. The single narrow-band signal is described by the product of a scalar time function s(t) with a time-invariant space vector S which is a vector of complex exponentials describing the relative phases of the incoming wave at the different sensors. The waveform s(t) is not of interest here and is dropped from the equations. Since the processor converges to a minimum output power state it will seek the unique weight vector W which minimises

$$P = W^T R W = W^T S S^T W \qquad (2.1)$$

Moreover except in the improbable event that S=C the output power will be zero unless the inequality in Eq. (1.2) becomes an equality. In other words, in the absence of background noise the purely linearly constrained processor will reject every signal (see for example Zahm[8]). The solution of Eqs. (1.2) with these assumptions can most easily be found by using the orthogonal eigenvectors of Hermitian matrix A as a coordinate system, i.e.

$$A \triangleq \sum_{1}^{N} a_i U_i U_i^T \quad (a_i \text{ real} \geqslant 0) \qquad (2.2)$$

where the orthogonal vectors U_i are normed to N, the number of sensors:

$$U_i^T U_j = N\delta_{(i-j)} \qquad (2.3)$$

*Assuming R is not singular

The vectors S, C, (norms = N) and W are resolved into weighted sums of the U_i:

$$S = \sum_1^N s_i U_i \qquad C = \sum_1^N c_i U_i \qquad W = \frac{1}{N} \sum_1^N w_i U_i \qquad (2.4)$$

Introducing Lagrange multipliers λ_w and λ_c we write Eq.(2.1) in the form

$$P = \left[\sum_1^N w_i^* \ s_i\right]^2 + \sum_1^N \left[\lambda_w(|w_i|^2 a_i - 1) + \lambda_c(c_i^* w_i - 1)\right] \qquad (2.5)$$

At the extrema of P the partial derivative with respect to each component w_i is zero, i.e.

$$\partial P/\partial w_i = 2s_i y + 2\lambda_w a_i w_i + \lambda_c c_i = 0, \ i = 1,N \qquad (2.6)$$

where y is the output amplitude

$$y = \sum_1^N s_i^* w_i \ \text{or} \ S^T W \qquad (2.7)$$

Since the component of W parallel to C is fixed by Eq.(1.2b) matrix A can be orthogonal to C. and this is accomplised by setting $U_1 = C$ and the corresponding coefficient $a_1 = 0$. From Eqs.(2.4) this immediately requires that $c_1 = 1$, $c_i = 0$, $i = 2,N$ and $w_1 = 1$. Eqs.(2.6) can be solved for λ_c when $i = 1$ and the remaining N-1 equations:

$$2s_i y + 2\lambda_w a_i w_i = 0, \ i = 2,N \qquad (2.8)$$

have the solutions

$$w_i = -e^{-j\theta} \ s_i a_i^{-1} \left[\sum_2^N |s_j|^2 a_j^{-1}\right]^{-\frac{1}{2}} \qquad i = 2,N \qquad (2.9)$$

or in actual variables, the component of W orthogonal to C is

$$\omega = W - \frac{1}{N} C = -e^{-j\theta} \ A^I S/(S^T A^I S)^{\frac{1}{2}} \qquad (2.10)$$

where A^I is the pseudo-inverse of A, $A^I = N^{-2} \sum_2^N a_i^{-1} U_i U_i^T$ (2.11)

and $\theta = \arg(S^T C)$ (2.12)

Finally the extrema of output amplitude $|y|$ are:

$$|y| = |s_1^* w_1| \pm \left[\sum |s_j|^2 a_j^{-1}\right]^{\frac{1}{2}} = |S^T C/N| \pm \frac{1}{N}\left[\sum |S^T U_i|^2 a_i^{-1}\right]^{\frac{1}{2}} \qquad (2.13)$$

The difference solution is used so long as the result is positive to find the minimum amplitude. If the difference goes negative it is assumed that the output amplitude is zero and inequality (1.2c) has slackened.

Fig. 3 shows the boundary of the feasible region corresponding to Eq.(1.2c) projected into the hyperplane orthogonal to C for a three element array with all real values. The minimum output power point occurs when the weight vector projection ω finds the point on the boundary such that the normal to the boundary is parallel but opposed to the projection of the signal ($P_C S = (I - CC^T/C^T C)S$). The concentric ellipses refer to a complex signal field condition and will be discussed later.

3. Time invariant sensor gain errors. The first design example is to make the processor robust for small sensor or channel gain errors. A signal arriving exactly from the look direction has vector $S \equiv C$ but in the presence of gain errors the processor receives a vector S' whose elements have amplitude and phase perturbations which can be represented by the addition of a vector E of random samples:

$$S' = S + \sigma E \tag{3.1}$$

Vector E is assumed orthogonal to S and is normed to N, $E^T E = N$ which allows σ to be identified as the sample standard deviation of the channel gains. It is assumed that while a likely upper bound for σ is known, the vector E is a random sample from a distribution which is isotropic in the vector space of the data (with the exception of direction S) and its direction is unknown. Writing $W = W_C + \omega$ where W_C is the conventional weight vector ($\frac{1}{N}C$) we have for the output amplitude of the adaptive system

$$y_a = (S + \sigma E)^T (W_C + \omega) = 1 + \sigma E^T \omega \tag{3.2}$$

which uses $E^T W_C = \omega^T W_C = 0$. From the triangle and Cauchy inequalities we have

$$1 - \sigma(\omega^T \omega E^T E)^{\frac{1}{2}} \leq |y_a| \leq 1 + \sigma(\omega^T \omega E^T E)^{\frac{1}{2}} \tag{3.3}$$

and for the single signal case the smaller value will prevail. Response variations limited to δ can be assured by the inequality

$$\sigma(\omega^T \omega E^T E)^{\frac{1}{2}} \leq \delta \tag{3.4}$$

which is easily converted to a bound on the norm of ω: $N\omega^T\omega \leq \delta^2/\sigma^2$. Thus the assumption of isotropic gain errors leads directly to a matrix A which is effectively an identity matrix and the constraint is thus similar to that used by Gilbert and Morgan[1]. Numerical values might be $\sigma = 0.1$ (7% amp. plus 4^o rms. phase error) and a limit of $\delta = 0.3$ which gives $N\omega^T\omega \leq 9$. Thus $A = \frac{N}{9} P_C$ or $a_i = \frac{1}{9}$, $i = 2,N$. Fig. 1 shows (a) conventional main lobe, (c) adaptive response with system under discussion and (d) adaptive system response with $A = |0|$. For the latter case the response is rather directional limited only by the incoherent noise background[8] which was set at -30 dB. The five element array gives a gain of

6.9 dB against this noise which is not bettered by the adaptive array. The effect of introducing channel gain errors corresponding to $\sigma = 0.1$ is shown in Fig. 2. The system under discussion (c) has lost about 4 dB of the signal while the system without the non-linear constraint (d) has lost about 18 dB so the desired robustness has been achieved.

An algebraic expression is easily derived for the adaptive system response from Eq. (2.13) in the absence of gain errors:

$$|y_a|_{min} \geq |y_c| - 3(1 - |y_c|^2)^{\frac{1}{2}} \quad (3.5)$$

which falls to zero when the normalised conventional response $|y_c| = |\frac{1}{N} S^T C|$ falls below $(0.9)^{\frac{1}{2}}$

4. Signal bearing mismatch. Figs. 1c and 2c show that the adaptive processor has a high resolution. This might be advantageous for some applications e.g. detection but for others e.g. communication systems might be regarded as a lack of robustness to signal bearing error. The matrix A can be modified to reduce the system sensitivity to bearing error as follows. A Taylor series expansion of the signal vector about the look direction (S=C) as a function of bearing has the form:

$$S(\theta + h) = S(\theta) + hS^{(1)}(\theta) + \frac{h^2}{2!}S^{(2)}(\theta) + \dots \quad (3.6)$$

where h is the bearing shift and $S^{(k)}(\theta)$ is a vector of the k-th derivatives of the elements of $S(\theta)$ with respect to bearing. For example, for a co-linear array lying on the x-axis with element coordinates x_i

$$S(\theta) = \exp(j2\pi x_i \sin(\theta)/\lambda)$$
$$S^{(1)}(\theta) = j2\pi/\lambda \; x_i \exp(j2\pi x_i \sin(\theta)/\lambda) \quad (3.7)$$

The derivative with respect to $\sin(\theta)$ has actually been used for simplicity. For small bearing shifts, $|h| \ll 1$, the change in signal vector is predominantly parallel to $S^{(1)}(\theta)$ and in Eq. (2.2) if we make $U_2 = N^{\frac{1}{2}} S^{(1)}(\theta)/||S^{(1)}(\theta)||$ then s_2 of Eq.(2.4) is proportional to h for small shifts and by using a large coefficient a_2 then from Eq.(2.13) the change in response caused by the bearing shift will be small* Numerical methods are called for at this point but as a rough approximation, if $a_2 = 1$ the resulting system will not null out a single signal until the conventional response has fallen by about 3dB. Matrix A has the remaining coefficients set to $\frac{1}{9}$ as before and the remaining orthonormal vectors U_i, i=3,N are an arbitrary set each orthogonal to U_2. This leads to

$$A = \frac{1}{9} NP_c + \frac{8}{9} U_2 U_2^T \quad (3.8)$$

*A technique much developed by LCDR A. Paulraj, Cochin Naval base, India.

Figs. 1b and 2b show the directional response using this matrix. It is interesting that the use of the modified constraint has caused a shift in the bearing estimation bias of the adaptive processor (taking maximum response as the source bearing). The processor with the 'spread' beam will not have such a good gain against isotropic noise as the original non-linear system in certain conditions and will certainly have worse performance for interferences close to the signal direction.

5. Complex signal and noise fields. The performance of the system in an environment rich in signals is difficult to analyse but simulations tend to show that the directional response is an approximate convolution of the single signal response with the true source spectrum. Thus in Fig. 4 there are six plane wave sources at bearings and with powers indicated and the array has six elements at 0.3λ spacing so is not very different from that used for Figs. 1 & 2. Channel gain errors of 10% were introduced for realism. The four curves are labelled as the previous figures and they indicate that the adaptive system with a simple norm constraint (c) has the best performance closely matched by (b) whose resolution is rather less. The conventional system has failed to detect the -20 dB source due to sidelobe spread while the purely linearly constrained adaptive system shows a strong suppression of the strongest source (d). Isotropic background noise is present at -30 dB and the adaptive processors' output falls to a level commensurate with this when steered between sources.

For an arbitrary sampled covariance matrix R the optimal non-linearly constrained weight vector W is given by the Lagrange method as $W = (R + \lambda A)^{-1}C/(C^T(R + \lambda A)^{-1}C)$ where the coefficient $\lambda (\geq 0)$ is chosen to satisfy Eq. (1.2c) but whose value is very difficult to determine except by eigenvector techniques and even then is markedly sensitive to numerical error. Non-linear programming methods[7] have been used to solve for the weight vector. The solution corresponds to the weight vector projection into the hyperplane orthonormal to C finding a point on the feasible region boundary where the gradient of the output power $\nabla_\omega(P)$ is normal to the boundary. The level curves of power in this projection are ellipsoidal and concentric at the point ω_o which is the projection of the purely linearly constrained optimal solution. These curves are shown in Fig. 3. The non-linearly constrained output power is that level curve which forms a tangent to the feasible region boundary.

6. Conclusions

The constrained processor described offers a robust adaptive estimate of angular power with limited a-priori knowledge and in the presence of random channel gain errors has much better performance than the linearly constrained adaptive processor. Robustness to bearing errors in one dimension has been discussed, this could easily be extended to two dimensions and insensitivity ro range error could be introduced by appropriate selection of constraint matrix A. Some caution is necessary since the system is still sensitive to large channel gain errors, e.g. sensor failure, and to correlated arrivals from different directions due to multipath effects. Greater robustness can be obtained by reducing the size of the feasible region[6] but null-steering performance begins to decline.

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7. References

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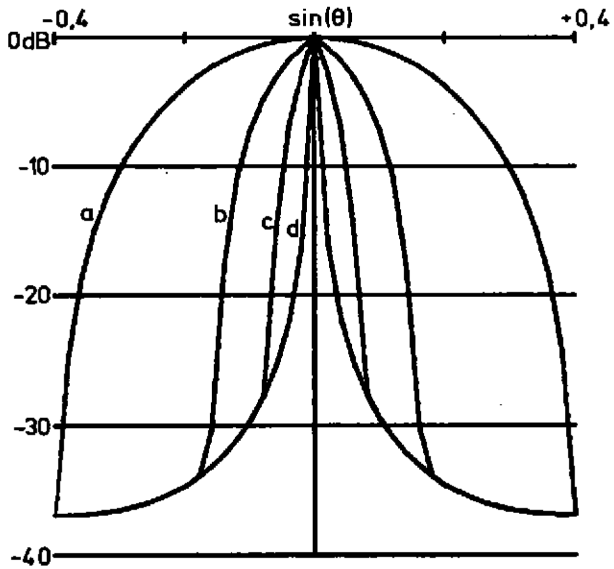

Fig. 1 5 element array main-lobe response.
(a) conv. (b) norm and tilt constr. (c) norm constr. only.
(d) linearly constrained.
Zero channel gain errors.

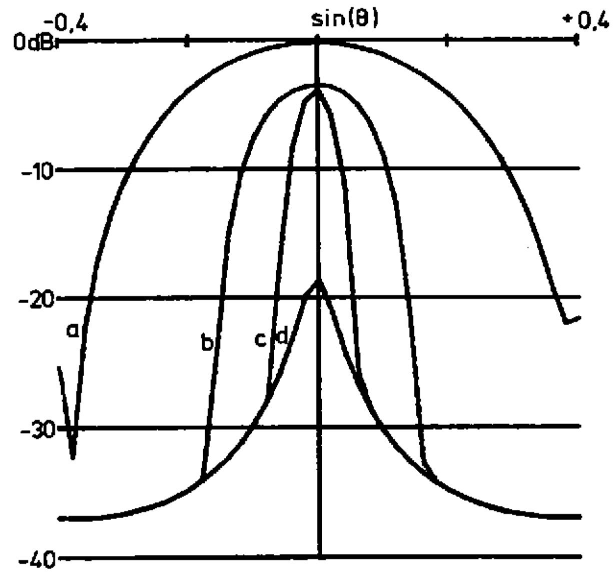

Fig. 2 5 element array main lobe response as Fig. 1 except 10% rms channel gain errors.

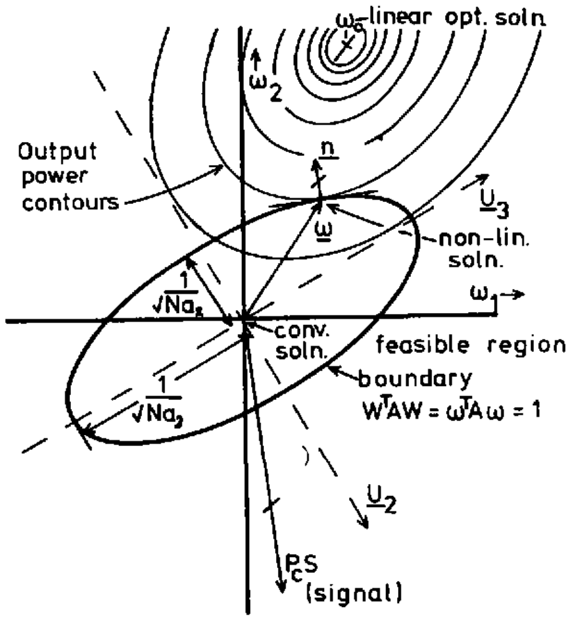

Fig. 3 Representation of feasible region and nature of conventional, linearly constrained and non-linearly constrained solutions.

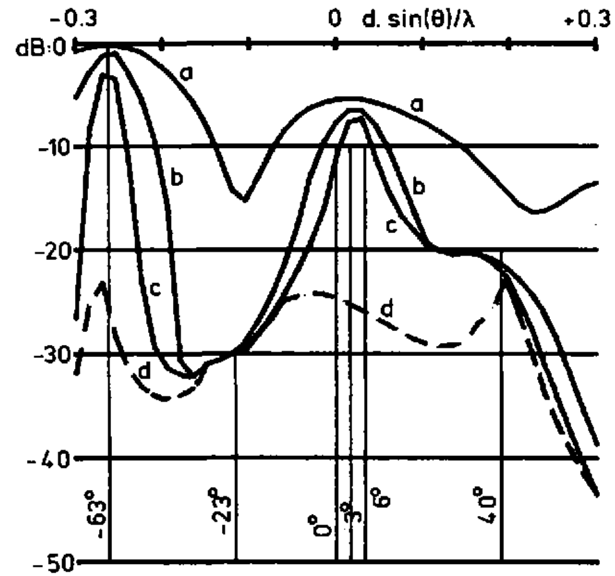

Fig. 4 Response of six element array to multiple sources. Legend as in Fig. 1.

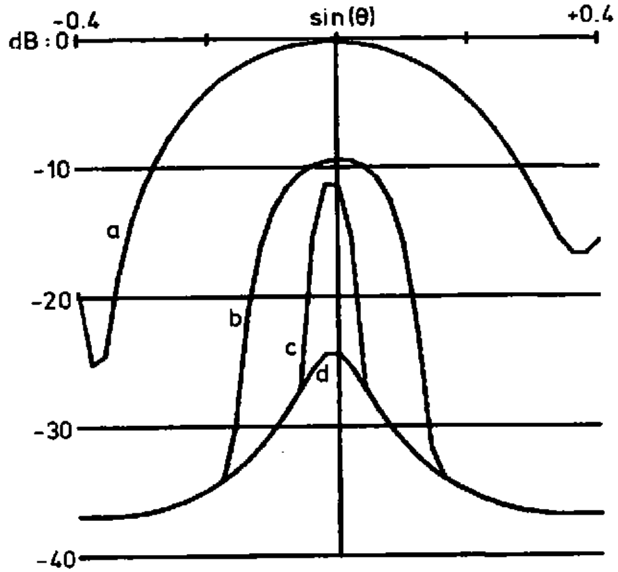

Fig. 1.5 Effect of 20% rms. channel gain errors.

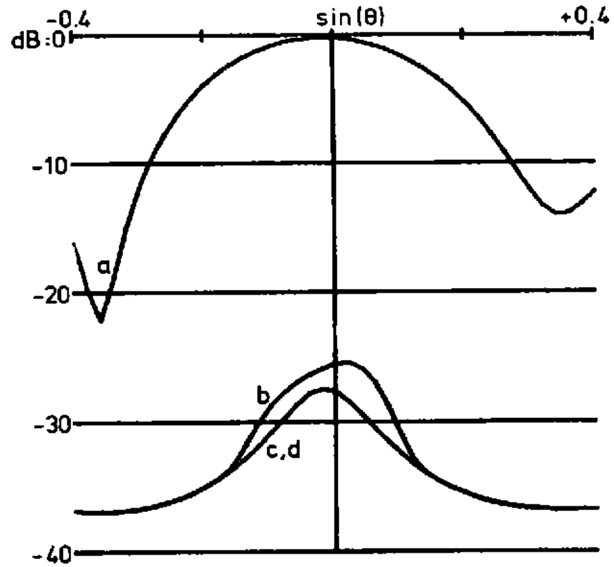

Fig. 1.6 Effect of 30% rms. channel gain errors.