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A CONTRIBUTION TO THE THEORY OF SCATTERING OF SOUND IN A STATISTICALLY INHOMOGENEOUS MEDIUM

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Introduction

In 1964 an article was published by Hufnagel and Stanley (1) in the Journal of the Optical Society of America, relating to the optical image-degrading effects of atmospheric turbulence. It was primarily concerned with the performance of astronomical telescopes. It was realized that the method that they used could apply also to the scattering of acoustic waves in water. The present paper is an attempt to explain the underlying ideas and to make them available to aeoustics engineers. As far as the authors are aware the principle involved, that of finding a statistical solution to the wave equation, has not appeared before in acoustical literatuee. Instead of setting out to find explicit expressions for the exact time-waveforms at various points in the medium, the idea is to establish a direct relationship between the time-averaged product of the fluctuations at a pair of receivers and the statistical properties of the medium. The mathematical analysis is somewhat involved and would be out of place here. It will merely be summarized. A fuller account will be found in reference 2.

Comparison with ray theory

According to ray theory, as a wave travels through each volume element of thickness its amplitude remains unchanged but its phase is advanced or retarded slightly by an amount proportional to the departure of the local velocity of propagation from its mean value. The total result therefore will be a phase shift which is obtained by summing the contributions from the volume elements which lie along the ray path. This simple approach is all very well provided the cumulative phase shift is very small. It fails to explain the amplitude changes which also occur and it soon breaks down completely when the phase fluctuations exceed a few degrees.

The important result which emerges from the Hufnagel and Stanley theory is that if we nevertheless pretend that the ray theory is exact it will lead to the correct result as far as the spatial coherence of the wave flucbuations are concerned. In other words, the coherence function of the real waves turns out to be just the same as that of the hypothetical waves which would be expected from simple ray theory. It should be pointed out however that their proof of this result depends on the assumption of plane waves and paralèll rays. Although fully justified in the astronomical case which was being considered this will not always be so in underwater acoustics. Although, as will be seen later, the authors have attempted to extend the theory to rays diverging from a point source and although this has led to plausible results, rigorous mathematical proof of the validity of the extension is still awaited. To start with, consider the case of parallel rays.

Application to pair of parallel rays

Suppose that the inhomogeneity of the medium is expressed by a quantity μ , which is a function of position and time, such that the velocity of propagation C is given by

$$\frac{1}{C} = \frac{1}{C_0}(1+\mu)$$

where C_0 is the mean value of C .

The total phase change along a path parallel to the z-axis is then $k\int \mu\, dz$ i.e. the wave is multiplied by a complex factor $e^{j\phi}$ where $\phi = k\int \mu\, dz$. The spatial coherence function is obtained by multiplying one such factor by the complex conjugate of that for the other path and is thus of the form

$$\langle e^{jks} \rangle$$

where $$S = \int_0^z (\mu_2 - \mu_1)\, dz$$

and $\langle\ \rangle$ denotes time-averaging

Making use of a theorem from probability theory, this can be transformed into the form

$$\langle e^{jks} \rangle = e^{-\frac{1}{2}k^2\langle s^2 \rangle}$$

The rest of the analysis is concerned with the evaluation of $\langle s^2 \rangle$. Before this can be done it is necessary to assumed some algebraic expression for the spatial coherence function of μ . A very convenient choice is the "Gaussian distribution" function which becomes

$$R(x) = \frac{\langle \mu_1 \mu_2 \rangle}{\langle \mu^2 \rangle} = e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}$$

where μ_1 and μ_2 are the values of μ at two points separated by a distance x and x_0 is the reference distance at which R has fallen to a value of $e^{-\frac{1}{2}}$. It then turns out that

$$\langle e^{jks} \rangle = e^{-K(1-\beta)}$$

where $$K = \sqrt{2\pi}\, k^2 z \langle \mu^2 \rangle x_0$$
$$\beta = e^{-\frac{1}{2}\left(\frac{a}{x_0}\right)^2}$$

and a is the distance between the two parallel paths.

Now the "signal" part of the wave can be identified by finding the limiting value to which $\langle e^{jks} \rangle$ tends as a is increased indefinitely. When this has been done the signal part of the wave can be subtracted to determine the "noise". This gives the coherence function for the noise as

$$\psi = \left(\frac{e^{\beta K}-1}{e^{K}-1}\right)$$

Finally we can use this result to compute, for various values of K, the value of the ratio (a/χ_0) at which ψ has fallen to e^{-1}; (see fig. 1).

The point to note is that when K is small, that is when the total fluctuation "noise" is small compared with the "signal", the coherence distance for the fluctuation noise is the same as that of μ, as predicted by simple theory (see, for example, ref. 3). When K is large however the coherence distance of the noise falls off at a rate inversely proportional to the range z and to the r.m.s. value of μ.

Divergent ray paths

The same general argument is used except that the separation a now becomes a variable and has to be treated at such when the various integration operations are carried out. The result is of the same form as before except that β becomes a little more complicated. It is now

$$\beta = \frac{\sqrt{\pi}}{2}\left(\frac{\operatorname{erf}\omega}{\omega}\right) \quad ; \quad \omega = \frac{1}{\sqrt{2}}\frac{a}{\chi_0}$$

One fact which may be of interest is that, when $K \ll 1$, the shape of the resulting spatial coherence curve for divergent rays happens to be very similar to that for parallel rays if χ_0 is replaced by $2\sqrt{2}\chi_0$. Thus, under these conditions, the coherence distance of the fluctuations is about three times as great for rays diverging from a point source as it would be for parallel rays which had travelled roughly the same distance. As stated earlier it is not known how much reliance can be placed on this result but it is certainly plausible, on the gounds that the average separation of divergent paths is less than that of parallel ones so that the waves would be expected to remain more coherent.

References

(1) R.E. Hufnagel & N.R. Stanley. "Modulation transfer function associated with image transmission through turbulent media". J.A.S.A., 54 p. 52, January 1964

(2) V.G. Welsby. (unpublished memorandum)

(3) V.G. Welsby. "Scattering phenomena in acoustic wave propagation" J. Sound Vib. 8, p. 64, 1968

See also: R.T. Aiken. "Propagation from a point source in a randomly refracting medium". B.S.T.J., May-June 1969, p. 1129

Conclusions

This analysis confirms that, for small amounts of fluctuation noise, the coherence distance of the noise is roughly equal to that of the medium itself. It goes further however and shows how the coherence of the noise falls when the noise-to-signal ratio passes a certain limit. The theory has been extended to the case of waves radiating from a point source embedded in the inhomogeneous medium. Although not rigorous mathematically this leads to a result which is at least plausible, in the sense that it predicts a greater coherence for waves from a point source than for plane waves which have travelled the same distance through the medium.

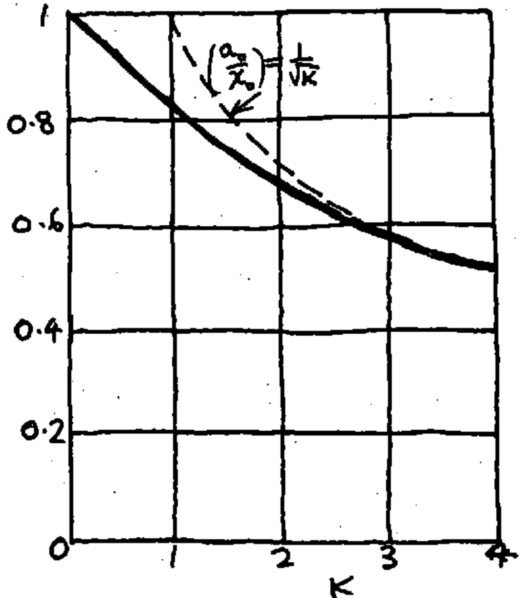

Fig. 1 Curve of $\left(\frac{a_0}{\chi_0}\right)$ versus K where $\left(\frac{a_0}{\chi_0}\right)$ is the value of $\left(\frac{a}{\chi_0}\right)$ for which $\psi = e^{-\frac{1}{2}}$

$$\psi = (e^{\beta K}-1)/(e^{K}-1)$$

and $\beta = e^{-\frac{1}{2}\left(\frac{a}{\chi_0}\right)^2}$