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NEAR-OPTIMUM SUPERRESOLUTION WITH A NUMERICALLY EFFICIENT ALGORITHM

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ABSTRACT

Statistical arguments are used to show that 3 generalised beams contain all the directional information required to resolve two close sources with Gaussian or sinusoidal waveforms with an array of arbitrary shape and number of elements. A Pisarenko/MUSIC type algorithm is used to estimate the source bearings in the three beams, the work load is very small, and performance approximates the Cramer-Rao Lower bound.

1. Introduction

The usefulness of high resolution is limited by two factors (a) the required SNR and signal purity (i.e. freedom from multipath and error) may not be available (b) the work load for the digital processing is very high, the typical algebraic technique requiring the eigenstructure of an N by N Hermitian matrix to be determined for an N element array. This paper describes a method for reducing the latter workload problem. First of all it is shown that if the sources lie in an angular region sufficiently small that a three-term Taylor series representation is an accurate description of their direction vectors then the projection of the data onto the low dimensional space spanned by the vector Taylor series is a sufficient statistic for estimation of the source parameters, viz. bearings and amplitudes. An algebraic method for processing this projection is then devised, based on established high resolution techniques. Since the correlation matrix is only 3x3 this has a small workload moreover the use of the Taylor series reduces the bearing estimation task from a trigonometric to a polynominal one eg./9/.

2. Mathematical Description.

Sufficient Statistics.

In an array of N elements let the direction vector for a source at bearing θ be $S(\theta)$, i.e. the elements of S are the received complex signals relative to a reference when the source is received with unit amplitude. When there are two signals present with complex amplitudes $\alpha_1(t)$ and $\alpha_2(t)$ the model for the received data is

$$X(t) = \alpha_1(t)S_1 + \alpha_2(t)S_2 + W(t) \qquad (1)$$

where W is a random noise vector with complex Normal $(0,\sigma^2)$ uncorrelated elements. If the amplitudes have a joint Gaussian distribution then X is Gaussian, however it will be convenient to place no prior restrictions on their statistics. The likelihood of observing T independent samples $X_1 \ldots, X_T$ is

$$p(X_1 \ldots X_T) = \frac{1}{(\pi\sigma^{2N})^T} \exp -\frac{1}{\sigma^2} \cdot \sum_1^T \| X_t - \mu_t \|^2 \qquad (2)$$

where μ_t is the mean value for sample t, and the log likelihood λ is proportional to

$$\lambda = \text{Const} - \sum_1^T \| X_t - \mu_t \|^2 \qquad (3)$$

The mean μ_t is given by $\mu_t = ZA_t$ where $Z \triangleq [S_1, S_2]$, and $A \triangleq (\alpha_{1t}, \alpha_{2t})^T$.

Maximisation of (3) over the domain of A with each sample allowed a different amplitude estimate gives the maximum likelihood form /2/

λ_{max} = Const -

$$\Sigma X_t^H (I - Z(Z^H Z)^{-1} Z^H) X_t \tag{4}$$

This result must be further maximised over the ranges of bearings of the two sources and normally this is a very difficult non-linear problem, especially at low SNR's /2,3/.

Taylor Series Approximation.

The likelihood (4) can be linearised by using a Taylor series approximation in the neighbourhood of the maximum likelihood point, this is an established statistical technique e.g. Cox /4/. Thus if a source lies at bearing $\theta+\delta$ we write

$$S(\theta+\delta) \simeq S(\theta) + \delta \dot{S}(\theta) + \frac{1}{2}\delta^2 \ddot{S}(\theta) = D\Delta$$

where $D \triangleq [S, \dot{S}, \frac{1}{2}\ddot{S}]$, with $\dot{S} = \partial S/\partial\theta$ etc. and $\Delta \triangleq (1,\delta,\delta^2)^T$. (5)

We now have

$$Z = D(\Delta_1, \Delta_2) = D\gamma \tag{6}$$

$$\mu = D\gamma A$$

and substituting into (3) obtain

$$\lambda = \text{Const} - \Sigma||X_t||^2 + \Sigma||D\gamma A||^2 - 2\mathrm{Re}\, \Sigma Y_t^H \gamma A_t \tag{7}$$

where $Y_t = D^H X_t$ is the three element statistic. The likelihood has factored into the form

$$\ell = \prod_1^T f(X_t) \cdot g(Y_t, \gamma A_t) \tag{8}$$

f being a function only of the data, and g being a function only of the statistic and the parameters and so the statistic is sufficient /4/ for the parameters of the Taylor series. The question of the accuracy of representation of the signals by this series is a separate non-statistical issue which is returned to later.

Subspace MUSIC method.

At this point it is necessary to determine the source parameters from the statistic. A simple Maximum likelyhood estimator (MLE) has not yet been found and the following modification of the MUSIC algorithm is suggested though it will show greater parameter estimate variance than a MLE. Statistical theory indicates that only the MLE can approach the Cramer-Rao lower bound and if another, ostensibly different, estimator approaches the bound then it too is a MLE and will give identical estimates sample by sample. Since any invertible function of a sufficient statistic is also sufficient we will begin by considering the projection of the estimated array covariance matrix R onto the subspace spanned by the derivative vectors D. Create an orthonormalised version of D by postmultiplying by an upper triangular matrix T: $Q=DT$ such that $Q^H Q=I_{3x3}$. The reduced rank matrix $B=QQ^H RQQ^H$ is the required data projection and desired sufficient statistic. To apply the MUSIC or Pisarenko method for the two sources the smallest eigenvector of B would be found, say U, and we would examine $|S(\theta)^H U|$ for minima : each corresponding to a source. This process can be transformed into an algebraic polynomial domain as follows. Write $S(\theta)^H = \Delta^H D^H$ from (5) and consider the 3x3 matrix $C=Q^H RQ$ with smallest eigenvector V. The matrices B and C are orthogonally similar and share the same eigenvalues, thus V is related to U by $U=QV$ and the cost for the Pisarenko method becomes

$$|S(\theta)^H U| = |\Delta^H D^H QV| \tag{9}$$

Writing the vector $D^H QV=F$ we have the equivalent polynomial $f_o+\delta f_1+\delta^2 f_2$.

This polynomial can be set to zero by equating δ to its roots, this determine the directions δ_i. This stage involves some approximation as the roots do not lie exactly on the real axis in the presence of noise so the result cannot be a MLE, however the accuracy is very good as the simulations below show. The matrix C is estimated by transforming the incoming vector samples via the matrix Q, thus the three term statistic from the antenna is the vector $G_t=Q^H X_t$ and C is estimated as

$\sum_1^T G_t G_t^H$. The first two numbers in G_t see Fig. 1) are equivalent to a radar monopulse antenna (or split-beam array) sum and difference beams while the third is a second derivative beam. A two target monopulse radar of this type is described by Peebles /6/,/7/ though his noise sources are introduced beam by beam in a way appropriate for separate dish feeds and the technique is not quite equivalent to the present noiseless beamforming with individual noises introduced element by element.

3. Numerical Tests.

The algorithm has been tested on an 18 element line array with uniform spacing for simplicity. A very similar algorithm has been shown to operate on non-uniform arrays, e.g. circular in a related coherent source application /8/ using single samples and this feature of its operation is not in doubt. The present tests demonstrate the bias introduced by the Taylor series expansion and test the parameter estimation variance against the Cramer-Rao lower bound.

In fig. 2 two equal power uncorrelated sources are placed symmetrically about broadside, there is no noise, and two vector samples are taken. The graph shows the estimated source separation vs. true separation. Clearly there is little significant bias until the sources approach one beamwidth separation and useable results are obtained out to three beamwidths separation. Beyond this catastrophic failure occurs even with no noise, while significant excess variance is expected if the source separation exceeds 1.5 beamwidths. This result is perfectly satisfactory.

In Fig. 3 two equal power sources are spaced by either half or a quarter beamwidth, four independent vector samples are taken to estimate C, and the variance of the estimated bearings is plotted as a function of SNR, the solid line showing the corresponding Cramer-Rao lower bound /1/. Again this result is very acceptable; the processor performance is virtually optimum.

4. Conclusions.

The tests on this linearised maximum likelihood estimation procedure have shown it to be numerically efficient and close to statistical optimality for close targets. Using it makes it feasible to apply approximate MLE to large arrays of arbitrary shape without incurring excessive computation cost since every array reduces to a three dimensional problem. The restriction to close targets must be observed and deleterious effects such as bias and excess variance are observed if the spacing is greater than one beamwidth. The way to operate this system in an unknown environment involves sequential decisions. First do a conventional beam scan to determine roughly what the angular spectrum is like. If only one peak is observed, possibly containing two unresolved targets, then the steering vectors D are expressed as a Taylor series about the centre of the peak and an attempt to resolve two sources in it is made. If there are two peaks resolved in the conventional scan, with greater than one beamwidth spacing, then they may be taken as the two required sources, or each may be individually probed with the high resolution system while steering a null to the other.

5. References.

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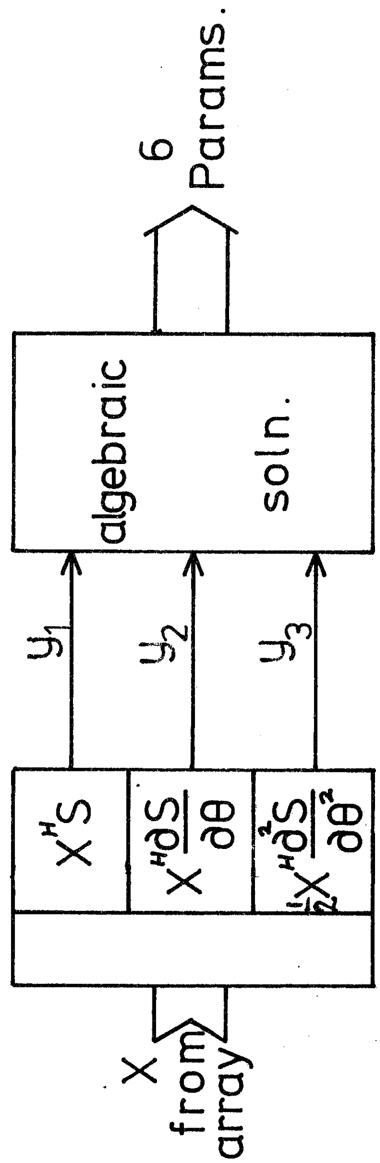

Fig. 1 Data reduction

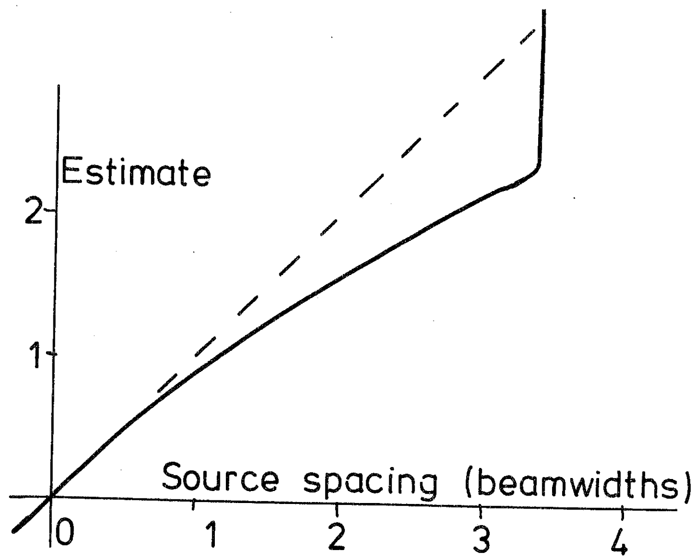

Fig. 2 Noise-free estimate bias

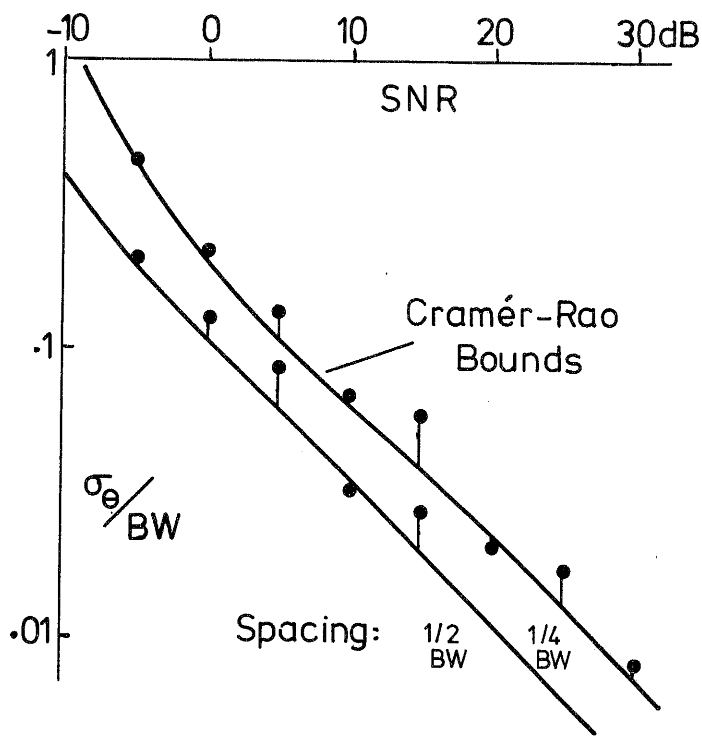

Fig. 3 Estimate variance