Proceedings of the Institute of Acoustics 'Spectral Analysis and its Use in Underwater Acoustics': Underwater Acoustics Group Conference, Imperial College, London, 29–30 April 1982

REMARKS ON THE STATISTICAL BEHAVIOR OF ORTHOGONAL BEAMFORMING

Johann F. Böhme

Lehrstuhl für Signaltheorie, Ruhr-Universität, 463o Bochum, W. Germany

ABSTRACT

Orthogonal beamforming is the name of certain high-resolution methods for estimating the spectra of a wave field received by an array of sensors. The methods use the eigenvalues and eigenvectors of the spectral matrix of the sensor outputs. The problem is to predict the behavior of such methods when only an estimate of the matrix is known. The sensor outputs may consist of sensor noise, ambient noise and noise from a finite set of discrete sources. The properties of the eigensystem of the spectral matrix in the case of weak ambient noise motivate the methods of orthogonal beamforming, for example Pisarenko's nonlinear peak estimates and the projection estimates of Owsley and Liggett. If the spectral matrix is estimated by one of the classical methods, some asymptotic distributional properties of the matrix estimate and its eigensystem are well known. They can be used to determine asymptotic expressions, e.g. for the first and second moments of the peak estimators and to approximate the distributions. The parameters, however, cannot be calculated in applications since the eigensystem of the exact spectral matrix is required. Therefore, we recently developed bounds for the deviation of the peak estimates which only use weak knowledge about the matrix. We applied some results on perturbations of hermitian operators. The asymptotic behavior of the bounds for the projection estimator is investigated and possibilities for their estimation are indicated. Finally, we report about extensive simulations with random matrices to evaluate the new bounds. As a result, we found that the projection estimator behaves stable and there are tight bounds if the eigenvalues of interest are sufficiently separated from the rest.

INTRODUCTION

Since more than ten years, high-resolution spectral estimation methods have been used for passive array processing in seismic and sonar applications. The methods take advantage of special propagation models of the wavefield. In particular, the surveillance of discrete sources motivates the use of coherent waves, for example plane waves for farfield sources. The spectral matrix of the sensor outputs, which are modeled by a stationary stochastic process, as the response to these waves has a simple structure. For estimating the propagation model, an analysis of the structure is usually done via an analysis of the eigenvalues and orthogonal eigenvectors of the spectral matrix which is assumed to be well estimated [5,11-16]. These methods are called orthogonal beamforming. In [11,12,14,16,18], possibilities are investigated for a direct identification of the steering vectors corresponding to the wavefronts incident from the discrete sources. We suppose that the propagation model is known except for some parameters, for example directions or wavenumbers, and we are interested in the behavior of diagrams over these parameters. The diagrams are interpreted like power spectra and can be compared with the corresponding classical beamforming spectra. We pick up well known methods [5,13,15] which we call peak estimates and try to predict their behavior by means of the behavior of the eigensystem of the estimated spectral matrix. The asymptotic statistical behavior of both, Pisarenko's nonlinear estimator [15] and Capon's high-resolution method [5], is known [6,15]. We show that of the estimators in [13,14]. Frequently in applications, the asymptotic expressions, e.g. for the moments cannot be calculated since the true spectral matrix and its eigensystem have to be known. We therefore developed bounds for deviation of the peak estimates computed from the estimated matrix to the best possible computed from the exact matrix. These bounds require less knowledge about the unknown matrix and seem to be more useful. We found them by application of results on perturbation of hermitian operators in [7]. The behavior of the bounds and a possibility for an estimation are discussed. Numerical experiments with random hermitian matrices were executed to show how good in the mean the bounds predict the deviations.

We remark that parts of the results described in this paper are contained in a technical report [2] and are summarized in [3].

The author is grateful to FGAN, 852o Wachtberg-Werthhoven, W. Germany, for partial support of the work presented herein. The author was assigned to FGAN in the period 1978 - 198o.

MODEL AND ESTIMATION OF SPECTRAL MATRICES

The array may consist of N identical omnidirectional sensors located at points x_n (n=O,...,N-1). The sensors have low pass character, and the sensor outputs are sampled in parallel with a period T. The model for the sensor outputs is a discrete stationary stochastic vector process with N components with expectation zero. The process is described by the spectral matrix $F(\lambda) = (f_{nj}(\lambda))$, where f_{nj} is the cross spectrum of the components from sensors at x_n and x_j, $\lambda=\omega T$ and ω is a frequency in cycles/s. Let us suppose similar to [12] that the process consists of three independent parts, nearfield noise, farfield noise or ambient noise and signals from K discrete sources in the farfield. Then, F is the sum of three non-negative hermitian matrices,

(1) $F(\lambda) = F_n(\lambda) + F_a(\lambda) + F_s(\lambda)$.

F_n corresponds to nearfield noise which is in the simplest case independent sensor noise such that $F_n = f_n I$, where f_n is the spectrum of sensor noise and I the unit matrix. F_a is due to ambient noise and has a representation

(2) $F_a(\lambda) = \int f_a(\lambda,\theta) D_\theta(\lambda) \overline{D_\theta(\lambda)} d\theta$,

where D_θ is the steering column vector for waves described by a parameter (or parameter vector) θ, the bar denotes transpose and conjugate-complex operation and $f_a(\lambda,.)$ is assumed to be a smooth spectrum. If we presume plane waves of a known velocity v of propation, ambient noise can be interpreted as a homogeneous random field and the parameters are bearing and elevation, D_θ is the direction vector with elements $\exp(i\omega\overline{u(\theta)} x_n/v)$ (n=o,...,N-1) and $-u(\theta)$ is a unit vector pointing in the direction of propagation of the wave. Finally, F_s describes the signals from K sources with parameters $\theta_o,\ldots,\theta_{K-1}$. If P is the matrix with column vectors D_{θ_k} (k=O,...,K-1) and G the KxK-matrix of cross spectra of signals received at the origin, then

(3) $F_s(\lambda) = P(\lambda)G(\lambda)\overline{P(\lambda)}$.

The rank of F_s is not greater than K which is the essential property. Similar signal models are also used in more general propagation situations, cf. [9]. Since F_s has rank(F_s) positive eigenvalues, the corresponding orthogonal eigenvectors are analysed to determine the steering vectors D_{θ_k} and the matrix G. Papers [12,14, 18] are references. We mention that a solution is not unique except for special cases.

Because of the noise components, the eigensystem of F cannot be used for a possible analysis as indicated above. We therefore assume either that we have some a priori knowledge about the coherence structure of noise and transform the data or that we can remove the nearfield component as described in [12]. Then, we investigate our problem with the assumption $F_n=f_nI$.

The spectral matrix F is unknown in applications and can be estimated from a long piece of the sampled outputs of all sensors if stationarity is assumed. The more difficult problem in the case of local stationarity is investigated in [11]. The estimate $\hat{F}$ has then to be analysed for an inference of the signal parameters θ_k etc. We suppose that $\hat{F}$ is determined by one of the classical methods, namely by the mean of periodogram matrices of succeeding streches of data or by smoothing the periodogram matrix of the data over frequencies. It is well known [4] that under some regularity conditions asymptotically, for large time-bandwidth products and small bandwidthes of the windows, the estimator $\hat{F}$ is normally distributed and its elements can be written

(4) $\hat{f}_{nj}(\lambda) \approx \int_{-\pi}^{\pi} w_B(\kappa) f_{nj}(\lambda-\kappa)d\kappa + Z_{nj}$.

Herein, w_B is a 2π-periodic function depending on the tapering of the data and in case of the smoothing window, and it behaves similar to the periodic version of Dirac's δ as the bandwidth parameter B approaches to zero. $Z_{nj} = \bar{Z}_{jn}$ are zero-mean random variables with covariances

(5) $Cov(Z_{nj}, Z_{kl}) \approx \frac{c}{BM} f_{nk}(\lambda) f_{jl}(-\lambda)$

if λ is not a multiple of π. The number c depends on the method, and BM is the time-bandwidth product. In connection with the problems discussed below, it is more convenient to approximate the distribution of $(BM/c)\hat{F}$ by a complex Wishart distribution with (BM/c) degrees of freedom and parameter matrix F. With respect to an analysis of $\hat{F}$, the smoothing effect in (4) is most problematic. We therefore have to control the bias carefully. One possibility is to choose a suitable frequency λ. We assume to have done this and shall omit the notation of λ mostly.

PEAK ESTIMATES

If the signals are incoherent, their spectral matrix F_s can be described similar to (2) with a discrete spectrum $f_s(\theta)$ which has δ-contributions for $\theta=\theta_k$ and wights proportional to the value of the spectrum of the signal k. For estimating the shape of the sum $f_a(\theta)+f_s(\theta)$ the diagram

(6) $\hat{g}(\theta) = \bar{D}_\theta \hat{F} D_\theta$

is used, sometimes modified by tapering the elements of D_θ, which corresponds to classical beamforming. The properties of this estimator in connection with the model of F are evident. The resolution of the method is limited by the geometry of the sensor field. If the steering vectors D_{θ_k} are well separated in this sense and there is no strong coherence between signals, the diagram will show a peak for $\theta=\theta_k$. The influence of sensor noise is charactarized by an additional bias which is constant for normalized steering vectors.

For improving the resolution, the peak estimators of orthogonal beamforming were introduced. They are functions of quadratic forms similar to (6), where $\hat{F}$ is replaced by a matrix which is a function of the eigensystem of $\hat{F}$. Let $\nu_o \geq \nu_1 \geq \ldots \geq \nu_{N-1}$ be the eigenvalues and $V_o, V_1, \ldots, V_{N-1}$ the corresponding orthonormal eigenvectors of $\hat{F}$. Pisarenko's nonlinear peak estimate [15] uses a monotonic function $h(x)>0$ for $x>0$ with inverse h^{-1} and is

(7) $\hat{g}_o(\theta) = h^{-1}(\bar{D}_\theta \Sigma_{j=o}^{N-1} h(\nu_j) V_j \bar{V}_j D_\theta)$.

For $h(x) = x$, we have (6) since the sum in (6) is then the dyadic decomposition of $\hat{F}$. If we use $h(x) = 1/x$ and $\hat{F}$ is the mean of periodogram matrices, we obtain Capon's high-resolution estimate [5]. Another class we call Owsley estimates is determined by a real interval which contains the eigenvalues $\nu_r, \dots, \nu_s$. First, we have the diagram

(8) $\hat{g}_1(\theta) = \bar{D}_\theta(\Sigma_{j=r}^{s} V_j \bar{V}_j) D_\theta = \Sigma_{j=r}^{s} |\bar{D}_\theta V_j|^2$,

where the sum in brackets is the projector into the linear space spanned by the orthonormal vectors $V_r, \dots, V_s$. A modification is to use the corresponding part of the dyadic decomposition of $\hat{F}$,

(9) $\hat{g}_2(\theta) = \bar{D}_\theta(\Sigma_{j=r}^{s} \nu_j V_j \bar{V}_j) D_\theta = \Sigma_{j=r}^{s} \nu_j |\bar{D}_\theta V_j|^2$.

If the eigenvalues of interest are either the smallest or the largest, we can consider (8) and (9) as limiting cases of $h(\hat{g}_o(\theta))$. In [11,13], $\hat{g}_1(\theta)$ is used for r=s=o which means the largest eigenvalue. Other authors used the smallest ones. The use of the smallest eigenvalues has the effect that the diagrams show valleys instead of peaks for $\theta = \theta_k$. Therefore, some authors [16,9] apply $1/\hat{g}_1(\theta)$.

ASYMPTOTIC BEHAVIOR

Since the following arguments do not depend on the parameter θ, its notation is omitted. We now discuss the asymptotic behavior of the diagrams (6) to (9) if $\hat{F}$ behaves asymptotically as described above. Starting from the asymptotic complex Wishart distribution, the results of Capon and Goodman [6] can be applied. Then we have that classical beamforming $\hat{g}$ is asymptotically distributed as $(\bar{D}FD)c/(2BM)\chi^2_{2BM/c}$. On the same way, one obtains for Capon's method that $\hat{g}_o = (\bar{D}\hat{F}^{-1}D)^{-1}$ behaves asymptotically as $(\bar{D}F^{-1}D)^{-1}c/(2BM)\chi^2_{2(BM/c-N+1)}$, where we have assumed that F has full rank and BM/c>N-1. Pisarenko [15] gives an argument that the estimator (7) is asymptotically normally distributed.

In the following we shall describe the behavior of the diagrams $\hat{g}_i(\theta)$ by means of the properties of the eigenvalues and eigenvectors of $\hat{F}$. Theorems 9.4.1 to 9.4.3 in [4] state that under some regularity conditions and if the eigenvalues of F are distinct, the eigenvalues and corresponding eigenvectors $\nu_o, \dots, \nu_{N-1}, V_o, \dots, V_{N-1}$ are asymptotically as $BM \to \infty$ and $B \to 0$ independently and normally distributed variables with

(1o) $\vec{\text{Ave}}\, \nu_j \doteq \mu_j$, $\vec{\text{Var}}\, \nu_j \doteq \mu_j^2 c/(BM)$,

$\vec{\text{Ave}}\, V_j \doteq U_j$, $\vec{\text{Cov}}(V_j, V_j) \doteq c/(BM)\Sigma_{1 \neq j} \mu_j \mu_1 (\mu_j - \mu_1)^{-2} U_1 \bar{U}_1$.

The notations $\vec{\text{Ave}}$ etc. mean expected values derived in a term by term manner from a Taylor expansion, and $\mu_o > \mu_1 > \dots > \mu_{N-1}$ are the eigenvalues and $U_o, \dots, U_{N-1}$ the corresponding eigenvectors of F. We assume that the biases of ν_j and V_j are small in comparison to variance terms. As a consequence, the variables $|\bar{D}V_j|^2$ are asymptotically distributed as $\sigma_j^2/2\chi_2^2(2|\bar{D}U_j|^2/\sigma_j^2)$, i.e. except for scaling non-central chi-squared with

$\vec{\text{Ave}}|\bar{D}V_j|^2 \doteq \sigma_j^2 + |\bar{D}U_j|^2$, $\vec{\text{Var}}|DV_j|^2 \doteq \sigma_j^2(\sigma_j^2 + 2|\bar{D}U_j|^2)$,

$\sigma_j^2 = \vec{\text{Var}}(\bar{D}V_j) \doteq c/(BM)\Sigma_{1 \neq j} \mu_j \mu_1 (\mu_j - \mu_1)^{-2} |\bar{D}U_1|^2$.

Now, it is simple to express the corresponding moments of the diagrams $\hat{g}_1$ and $\hat{g}_2$. For (8), we find

(11) $\vec{\text{Ave}}\hat{g}_1 \doteq g_1 + \Sigma_{j=r}^{s} \sigma_j^2$,

where g_i denotes the best possible diagram which is defined so that in the definition of $\hat{g}_i$ the eigensystem of $\hat{F}$ is replaced by that of F, for example

$g_1 = \Sigma_{j=r}^{s} |\bar{D}U_j|^2$,

(12) $\mathrm{Var}\, \hat{g}_1 \doteq 2\Sigma_{j=r}^{s} \sigma_j^2 |\bar{D}U_j|^2$.

We have omitted terms $(BM)^{-k}$ for k>1. The moments of (8) are

(13) $\mathrm{Ave}\,\hat{g}_2 \doteq g_2 + \Sigma_{j=r}^{s} \mu_j \sigma_j^2$,

(14) $\mathrm{Var}\,\hat{g}_2 \doteq \Sigma_{j=r}^{s} \mu_j^2 |\bar{D}U_j|^2 (2\sigma_j^2 + c/(BM)|\bar{D}U_j|^2)$.

A computation of the moments of $\hat{g}_o$ in a similar way does not result in a good approximation since we did not take into account the first order errors in Ave $\hat{\mu}_j$ and Ave $\vec{V}_j$. Therefore, we note the following expression which can be obtained on the same way as shown in [15],

(15) $\mathrm{Ave}(\hat{g}_o - g_o)^2 \doteq c/(BM) h'(g_o)^{-2} \Sigma_{j,l=o}^{N-1} \mu_j \mu_l (h(\mu_j) - h(\mu_l))^2 / (\mu_j - \mu_l)^2 |\bar{D}U_j|^2 |\bar{D}U_l|^2$,

where h' is the first derivative of h which has to be a holomorphic function in the right half plane and $(h(\mu_j)-h(\mu_l))/(\mu_j-\mu_l)$ is replaced by $h'(\mu_j)$ for $\mu_j = \mu_l$.

In the last paragraph, we presumed distinct eigenvalues of F. If some eigenvalues are equal, for example in model (1) the smallest eigenvalues when $F_a = 0$ and $F_n = f_n I$, the corresponding eigenvectors V_j of $\hat{F}$ have not the asymptotic properties (1o). However, in Pisarenko's arguments this assumption is not used, and (15) can be further applied. As indicated in the last chapter, (8) and (9) can be approximted by $h(\hat{g}_o)$ if for example the smallest eigenvalues are of interest and h is a suitable holomorphic function. Then, we obtain asymptotic expressions for the mean square errors similar to the right-hand side of (15) except for the factor $h'(g_o)^{-2}$. The problem in this connection is that we have to know which eigenvalues are equal and then to choose the integers r and s in (8) and (9) appropriatly. When for example in (1) $F_a = 0$ and $F_n = f_n I$, we have to estimate the rank of F_s. In [11], a solution for this case is given. For testing the equality of eigenvalues, we refer to the statistical literature e.g. [1o],[17]. If $\mu_r = \ldots = \mu_s$ and r and s are extremum, the interesting point is the asymptotic behavior of $\Sigma_{j=r}^{s} |DV_j|^2 = \bar{D}(\Sigma_{j=r}^{s} V_j \bar{V}_j) D$. One can approach this by use of papers [1], [8].

BOUNDS

Since in applications the exact spectral matrix F and its eigensystem are unknown, the asymptotic expressions (11) to (15) cannot be evaluated. What we can say in a concrete situation is that $\hat{g}_1$ and $\hat{g}_2$ tend to overestimate g_1 and g_2 and that biases and variances of the $\hat{g}_1$ are of order $(BM)^{-1}$ as $BM\to\infty$. We therefore developed bounds for the deviations

(16) $d_i = |\hat{g}_i - g_i|$

from the peak estimate $\hat{g}_i$ to the best possible diagrans g_i which require less information about F.

We first investigate $d_i = |\bar{D}A_i D|$ for i = 1,2, where $A_1 = Q_{rs} - P_{rs}$,

$Q_{rs} = \Sigma_{j=r}^{s} V_j \bar{V}_j$, $P_{rs} = \Sigma_{j=r}^{s} U_j \bar{U}_j$ and $A_2 = \Sigma_{j=r}^{s} (\hat{\mu}_j Q_{jj} - \mu_j P_{jj})$ are hermitian matrices.

Using the vector norm $\|D\| = (\bar{D}D)^{1/2}$, we have for i = 1,2

(17) $d_i \le \|D\|^2 \delta_i$,

$\delta_i \le \sup_\theta d_i(\theta)/\|D_\theta\|^2 \le (\text{max eigenvalue } (\bar{A}_i A_i))^{1/2} = \|A_i\|_1$.

This bound cannot be improved in general. The matrix norm $\| A_i \|_1$ is unitary-invariant, i.e. $\| BA_iC \|_1 = \| A_i \|_1$ for unitary matrices B and C. Every such matrix norm $\| A_i \|$ with $\| A_i \|_1 \leq \| A_i \|$ can be used to bound δ_i, for example the square norm $\| A_i \|_{sq} = (tr(\bar{A}_i A_i))^{1/2}$.

As shown in [2], bounds for $\| A_1 \|$ were found by application of Davis' and Kahan's results on weak perturbation of hermitian operators [7]:
Assuming $2(s-r+1) \leq N$ and the existence of a number $\Delta > 0$ so that

- the eigenvalues $\nu_o \geq \ldots \geq \nu_{N-1}$ of $\hat{F}$ satisfy $\nu_s - \nu_{s+1} \geq \Delta$ if $s < N-1$ and $\nu_{r-1} - \nu_r \geq \Delta$ if $r > 0$,
- the eigenvalues $\mu_o \geq \ldots \geq \mu_{N-1}$ of F satisfy $\mu_s - \nu_s \leq \Delta/2$ and $\nu_r - \mu_r \leq \Delta/2$,
- $\| H \|_1 \leq \Delta/2$, where $H = \hat{F} - F$ is called perturbation,
  then we have

(18) $\| A_1 \| \leq \| Q_{rs} - P_{rs} \| \leq 2^{1/2} \| H \| / \Delta$

and for $\| \ \|_1$ the more exact bound

(19) $\| A_1 \|_1 \leq 2^{1/2} (1-(1-4 \| H \|_1^2 / \Delta^2)^{1/2})^{1/2}$.

Consequently, if $\hat{F}$ is a good estimate of F and if the eigenvalues of interest are well separated from the other eigenvalues of $\hat{F}$ by a number $\Delta > 0$ which we know, then $d_1 / \| D \|^2$ is bounded by the right-hand sides of (18) and (19). In model (1), the K largest eigenvalues are well separated from the other ones if the weakest signal has sufficient signal-to-noise ratio, the coherence between the sources is low and there is only weak coherent noise.

To find bounds for d_2, we write

$\| A_2 \| = \| \Sigma_{j=r}^{s} (\nu_j (Q_{jj} - P_{jj}) + (\nu_j - \mu_j) P_{jj}) \|$

if the μ_j are distinct and obtain

(2o) $\| A_2 \| \leq \Sigma_{j=r}^{s} (\nu_j \| Q_{jj} - P_{jj} \| + |\nu_j - \mu_j|)$.

Similarly, if $\mu_r = \ldots = \mu_s$,

(21) $\| A_2 \| \leq \Sigma_{j=r}^{s} |\nu_j - \mu_j| + (\nu_s + |\nu_s - \mu_s|) \| Q_{rs} - P_{rs} \|$.

If we bound $|\nu_j - \mu_j| \leq \| H \|_1 \leq \| H \|$ and use (18) or (19) we get bounds for $\| A_2 \|$, e.g. by (19) and (2o)

(22) $\| A_2 \|_1 \leq \Sigma_{j=r}^{s} (\| H \|_1 + \nu_j 2^{-1/2} (1-(1-4 \| H \|_1 / \Delta_j^2)^{1/2})^{1/2})$,

where Δ_j is the distance from ν_j to the nearest eigenvalue of $\hat{F}$. As known, we may assume that with probability one all $\Delta_j > 0$ if $(BM)/c > N$ and zero is not an eigenvalue of F.

In analogy, we can obtain bounds for

$d_h = |h(\hat{g}_o) - h(g_o)| \leq \| D \|^2 \| A_h \|$, $A_h = \Sigma_{j=o}^{N-1} (h(\nu_j) Q_{jj} - h(\mu_j) P_{jj})$,

when in (2o) and (21) ν_j is replaced by $h(\nu_j)$ etc. If h is specified, we find bounds for $|h(\nu_j) - h(\mu_j)|$ depending on h, ν_j and $\| H \|$ which results in bounds for $\| A_h \|$. Finally, $d_o = |h^{-1}(h(\hat{g}_o)) - h^{-1}(h(g_o))|$ is treated like $|h(\nu_j) - h(\mu_j)|$.
As an example, in [3] is noted the corresponding bound for Capon's method. Space doesn't allow to go into the details. We only remark that the bounds composed in this way are not tight in contast to (18). Possibly, an application of the approximation method in the proof of Theorem 1 in [15] gives better bounds.

The only unknowns of the bounds (18), (19) and (22) are the norms of $H = \hat{F}-F$. The behavior of these norms and possibilities for their estimation are of interest in applications. First, we apply Theorem 7.7.3 in [4] which states that under some conditions asymptotically for $BM\to\infty$ and $B\to 0$ the elements $\hat{f}_{nj}(\lambda)$ of $\hat{F}(\lambda)$ satisfy

$$\sup_\lambda |\hat{f}_{nj}(\lambda)-E\hat{f}_{nj}(\lambda)| = O((BM/\ln(1/B))^{-1/2})$$

with probability 1. Since we can write

$$\sup_\lambda |E\hat{f}_{nj}(\lambda)-f_{nj}(\lambda)| = O(B)+O((BM)^{-1}),$$

we obtain asymptotically

$$\|H\|_1 \le \|H\|_{sq} = O(B) + O((BM/\ln(1/B))^{-1/2})$$

with probability 1 and error terms being uniform in λ. In comparison with this result, we know

$$E\|H\|_1 \le E\|H\|_{sq} \le (E\|H\|_{sq}^2)^{1/2} = O(B_M)+O((B_M M)^{-1/2}).$$

The estimation of $\|H\|$ and especially of $\|H\|_1$ in applications from data or from $\hat{F}$ is not yet solved satisfactory. For $\|H\|_{sq} = (\Sigma_{n,m}|\hat{f}_{nm}-f_{nm}|^2)^{1/2}$ there is the crude estimate $(c/BM)^{1/2}\mathrm{tr}(\hat{F})$. Here, we have used that $\hat{F}$ is an estimate (4) with low bias and covariance (5).

NUMERICAL EXPERIMENTS

For evaluating the bounds (19) and (22), numerical experiments were executed by means of an FPS array processor 12o B. The model (1) was used for a line array with N=8,12,16 sensors having a mutual distance of a half wavelength. We used two noise situations a) sensor noise and isotropic ambient noise of the same power, b) ambient noise alone and K=1,2 incoherent signal sources with bearings $\theta_o=1o^o$ and $\theta_1=4o^o$ and powers $g_{oo}=1$ and $g_{11}=o.5$, respectively. (Our problem was not to investigate the resolution of the methods.) For signal-to-noise ratios (S/N) = ∞,6,3,o-3,-6,-1odB, the best possible diagrams $g_1(\theta)$ and $g_2(\theta)$ showed the peaks correctly as decreasing (S/N) up to -6dB. We investigated the behaviors of the deviations only up to this value.

Estimates $\hat{F}=(\hat{f}_{nj})$ were simulated by use of (4) assuming no bias and a simplified structure of independent, zero-mean triangular distributed pseudo-random variables Z_{nj} in a real representation with variances as given by (5). For scaling parameters $a=(6c/(BM))^{1/2}$ = .o1,.1,.2,.3,.5,.7,1.o,1.3 and each possible matrix F their estimates $\hat{F}$ were computed, then the diagrams $g_i(\theta)$ and $\hat{g}_i(\theta)$ (i=1,2), the numbers δ_1 and δ_2 as global deviations and finally the bounds (19) and (22). For comparing the different situations, we asked for the factor by which a bound is greater than the corresponding global deviation. We found that in comparable situations in the mean the bound (19) is 2 to 3 times greater than the global deviation δ_1 and (22) about 1.5 KM-times greater than δ_2.

Concluding, if there is a good estimate of $\|H\| = \|\hat{F}-F\|$ and if the eigenvalues of interest are well separated from the other eigenvalues of $\hat{F}$, then the bounds (18) and (19) give good tools to predict the behavior of Owsley's projection estimate $\hat{g}_1$. The bound (22) seems to be crude and should be improved.

REFERENCES

1. Anderson, T.W.: Asymptotic Theory for Principle Component Analysis, Ann. Math. Statist. 34(1963), 122-148
2. Böhme, J.F.: Empfindlichkeitsuntersuchungen für orthogonales Beamformen, Forschungsbericht Nr. 1-81 des Forschungsinstituts für Hochfrequenzphysik der FGAN, Wachtberg-Werthhoven, W. Germany, 1981
3. --: On the Sensitivity of Orthogonal Beamforming, Proc. IEEE ICASSP, Paris, 1982
4. Brillinger, D.R.: Time Series: Data Analysis and Theory, Holt, Rinehart and Winston, New York, 1975
5. Capon, J.: High-Resolution Frequency-Wavenumber Spectrum Analysis, Proc. IEEE 57 (1969), 14o8-1418
6. Capon, J. and Goodman, N.R.: Probability Distributions for Estimators of the Frequency-Wavenumber Spectrum, Proc. IEEE 58 (197o), 1785-1786
7. Davis, C. and Kahan, W.M.: Some New Bounds on Perturbation of Subspaces, Bull. Am. Math. Soc. 75 (1969), 863-868
8. James, A.T.: Tests for a Prescribed Subspace of Principle Components, in Multivariable Analysis IV, P.R. Krishnaiah ed., North-Holland Publ., Amsterdam, 1977, 73-77
9. Klemm, R.: Use of Generalized Resolution Methods to Locate Sources in Random Dispersive Media. IEE Proc. 127 (198o) Pt. F No. 1
10. Krishnaiah, P.R. and Lee, J.C.: Inference on the Eigenvalues of the Covariance Matrices of Real and Complex Multivariable Normal Populations, in Multivariate Analysis IV, P.R. Krishnaiah ed., North-Holland Publ., Amsterdam, 1977, 95-1o3
11. Liggett, W.S.Jr.: Passive Sonar: Fitting Models to Multiple Time Series, in Signal Processing, J.W.R. Griffith et al. eds., Academic Press, New York, 1973, 327-345
12. --: Passive Sonar Processing for Noise with Unknown Covariances Structure, J. Acoust. Soc. Am. 51 (1972), 24-3o
13. Owsley, N.L.: A Recent Trend in Adaptive Spacial Processing for Sensor Arrays: Constrained Adaption, in Signal Processing, J.W.R. Griffith et al. eds., Academic Press, New York, 1973, 591-6o4
14. --: Spectral Signal Set Extraction, in Aspects of Signal Processing, G. Tocconi ed., D.Reidel Publ., Dordrecht-Holland, 1977, 469-475
15. Pisarenko, V.F.: On Estimation of Spectra by Means of Nonlinear Functions of the Covariance Matrix, Geophys. J. Roy. Astron. Soc. 28 (1972), 511-531
16. --: The Retrieval of Harmonics from a Covariance Function, Geophys. J. Roy. Astron. Soc. 33 (1973), 347-366
17. Priestley, M.B., Subba Rao, T. and Tong, H.: Identification of the Structure of Multivariable Stochastic Systems, in Multivariate Analysis III, P.R. Krishnaiah ed., Academic Press, New York, 1973, 351-368
18. Reddi, S.S.: Multiple Source Location: A Digital Approach, IEEE Trans. AES15 (1979), 95-1o5