Stochastic Matched Filter

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Abstract

In this paper, we shall look into the problem of detecting a signal belonging to a family of signals $\{S_j\}$. This family is defined by its statistical properties. The caracteristics of the interfering noise are also known. We assume that each signal S_j is defined on a close interval (0,T). Outside this interval, S_j is 0.

1- Chose of an optimisation criteria

1-1 We shall show that in order to use all the information that is available a priori, it is necessary to construct a serie of filters whose properties are comparable to those of a matched filter. We shall present a practical method of calculating the filters, and show that there exists a basis in which both noise and signal components are uncorrelated random variables.

1-2 Definition, notations and link with matched filtering.

$S_1(t)$, a deterministic signal, is 0 ouside (0,T).
$X(t)$ is the noise, whose autocorrelation function is $R_{xx}(t)$. In the case of matched filtering, we assume the signal of interest is either absent, or equal to $S_1(t)$, and this in presence of noise.

Consequently if the signal is present, the delay τ is necessarily 0. We define on (0,T) a function A(t) such that the ratio K is maximized:

$$K = \frac{\left| \int_T S_1(t) A(t)\, dt \right|^2}{E\left\{ \left| \int_T A(t)\, X(t)\, dt \right|^2 \right\}} \qquad (1)$$

The optimal function A(t) is the response of the matched filter.

1.3 Case of an unknow delay τ.

When the delay τ is not zero, it is necessary to replace $S_1(t)$ by the random function $\{S_1(t-\tau)\}$, where τ is a random variable that we may assure, in absence of additional information, to be uniformaly distributed on a physically acceptable interval.

It is logical to redefine the ratio K as

$$K = \frac{E\left\{ \left| \int_T S_1(t-\tau)\, A(t)\, dt \right|^2 \right\}}{\iint_{T*T} A(t) R_{xx}(t\,,m) A^*(m)\, dt\, dm} \qquad (2)$$

The numerator of this fraction can be written

$$\iint_{T*T} A(t) E_\tau\{ S_1(t-\tau)\, S_1(m-\tau) \}\, A^*(m)\, dt\, dm$$

or equivalently

$$\iint_{T*T} A(t)\Gamma_{S_1}(t-m)\,A^*(m)\,dt\,dm \qquad (3)$$

1-4 Case of Dopplerised signal.

Instead of considering the random function $\{S1(t-\tau)\}$, where τ is uniformely distributed random variable, it is preferable to consider $\{S1(k(t-\tau))\}$ where k is a Doppler coefficient. k is also a random variable independant τ, uniformely distributed in the interval (k1,k2).

The corresponding autocorrelation function to this new problem will be:

$$\Gamma(t-m) = E_k\{E_\tau\{\, S_1(\vec{k}(t-\tau))\, S_1^*(\vec{k}(m-\tau))\}\}$$

In an even more general approach, if the random function is of the form $S_1(\vec{k}, t-\tau)$ the corresponding autocorrelation function will be:

$$\int_{D(k)} p(\vec{k})\; E_\tau\{\, S_1(\vec{k}, t-\tau)\, S_1^*(\vec{k}, m-\tau)\}\, d\vec{k}$$

where $D(\vec{k}) \subset R^N$ and $p(\vec{k})$ is the probability density of $\vec{k}$.

2 Optimisation of the signal to noise ratio

2-1 We must find the function A(t) such the quantity

$$K = \frac{\iint_{T*T} A(t)\,\Gamma_{S_1}(t-m)\,A^*(m)\,dt\,dm}{\iint_{T*T} A(t)\,R_{xx}(t,m)\,A^*(m)\,dt\,dm} \qquad (4)$$

is maximum.

In order to optimize K according to equation (4), we shall discretize the problem.
Let $t_1, t_2, \ldots, t_N$ instants distributed along (0;T).
Let

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad X = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_N) \end{bmatrix}, \quad A^t = [\,a_1^*\; a_2^* \ldots a_N^*]$$

In the discret case, the equivalent of the signal to noise ratio is:

$$K=\frac{\sum_{1}^{N}\sum_{1}^{N} a_{\lambda}\Gamma_{s_1}(t_{\lambda}-t_{\mu})a_{\mu}^{*}}{\sum_{1}^{N}\sum_{1}^{N} a_{k}R_{xx}(t_{k}-t_{\lambda})a_{\lambda}^{*}}$$

Let Γ, the symetrical matrix of elements:

$$\Gamma_{s_1}(t_{\lambda}-t_{\mu})$$

and R the noise covariance matrix, then K can be rewritten as

$$K=\frac{A^{t}\Gamma A}{A^{t}RA} \qquad (5)$$

2-2 Solutions and interpretations.

a) We see that K is a generalised Rayleigh quotient. It follows that K will be the largest possible if A is the eigenvector associated to the largest value of the matrix $C=R^{-1}*\Gamma$

The square matrix C is assumed to have N distinct eigenvalues and N eigenvectors.
C, the product of two symetric square matrices, is not itself symetric.
The matrices C^t and C have the same eigen values.
Let

$C X_i=\lambda_i X_i$,
$C^t Y_j=\lambda_j Y_j$,

We know that

$$Y_j^t X_i=0 \qquad \text{if} \quad i\neq j \qquad (6)$$

The relation (6) expresses the fact that the left and right eigenmodes are orthogonal.

b) Let $\qquad Y_j= R X_j \qquad$ (7)

Whe shaw that Y_j is a left eigenmode of C.
By definition $R^{-1}\Gamma X_j=\lambda_j X_j$.
Since Γ and R are symetrical, by transposition we have

$$X_i^t\Gamma=\lambda_j X_j R ,$$

or, with (7)

$Y_j^t R^{-1}\Gamma=\lambda_j Y_j^t$.
$Y_j^t C=\lambda_j Y_j^t$.
$C^t Y_j=\lambda_j Y_j$,

We shall have $X_i^t R X_j=0 \qquad \text{if} \quad i\neq j \qquad (8)$

We chose the X_i such that $X_i^t R X_i = 1$. (9)

c) Physical interpretation .

Let us suppose that in the expression

$$K = \frac{A^t \Gamma A}{A^t R A},$$

we set $A=X_0$;

We obtain

$$K = \frac{X_0^t \Gamma X_0}{X_0^t R X_0} = X_0^t \lambda_0 R X_0 = \lambda_0 .$$

The highest eigenvalue represents the optimal signal to noise ratio.

3 Decomposition of the noise and signal along the basis vectors Y_i.

3-1 Decomposition of the noise.

Let $B = \Sigma b_i Y_i$,

Evaluating bi and its variance:

$$X_i^t B = X^t{}_i \Sigma b_j Y_j = \Sigma b_j X_i^t Y_j = b_i X_i^t Y_i = b_i X_i^t R X_i = b_i.$$

$$E\left\{|b_i|^2\right\} = E\left\{X_i^t B B^t X_i\right\} = X_i^t E\left\{B B^t\right\} X_i = X_i^t R X_i$$

$$E\left\{b_i b_j^*\right\} = E\left\{X_i^t B B^t X_j\right\} = X_i^t R X_j = 0.$$

conclusion1: if we decompose the noise along the basis vectors $Y_i = RX_i$, the coefficients of the decomposition are bi uncorrelated variables of power equal to1.

3-2 Decomposition of a centered signal along the basis Y_i .

Let $S = \Sigma s_i Y_i$.

We shall successively have

$$X_i^t S = X_i^t \, \Sigma s_j Y_j = \Sigma s_j X_i^t Y_j = s_i X_i^t R X_i,$$

$$s_i = X_i^t S.$$

The power is given by

$$E\left\{X_i^t S S^t X_i\right\} = X_i^t \Gamma X_i = \lambda_i.$$

Also

$$E\left\{X_i^t S S^t X_j\right\} = X_i^t \Gamma X_j = X_i^t \lambda_j R X_j = 0.$$

Conclusion2: The decomposition of the signal along the basis functions Yi are si uncorrelated variables of power equal to λ_i .

4 Tentative regrouping of the information output by the different filters.

The optimisation of the signal to noise ratio K, has led us to calculate the optimal filter X_0, and also others filters $X_1, X_2, \ldots, X_{N-1}$.
It is tempting to regroup the all the information from these different filters to obtain a more synthetic information.
Let us assume the signal to be stationary centered gaussian noise of covariance matrice Γ, and the interfering noise to also be gaussian of covariance matrix R.

Using the basis vectors Y_i, the covariance matrix of the noise is $R = I_{NxN}$
Assuming the noise to be independant of the signal, the covariance matrix of the signal added to noise is

$$D = \begin{bmatrix} 1+\lambda_0 & & 0 \\ 0 \;\; 1+\lambda_1 & & 0 \\ 0\,0 & & 1+\lambda_{N-1} \end{bmatrix}$$

An hypothesis test leads us to compare the following two laws:

$$\frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{D}} \exp(-\tfrac{1}{2} Z^t D^{-1} Z) \qquad\qquad \frac{1}{(2\pi)^{\frac{N}{2}}} \exp(-\tfrac{1}{2} Z^t Z)$$

In other words, we use the quantity T which is the logarithm of the likelyhood ratio.

This is written $T = \Sigma z_i^2 \frac{\lambda_i}{1+\lambda_i}$ where z_i corresponds to the filtering of corrupted signal by the different X_i filters.

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